A CONCISE HISTORY OF MATHEMATICS

STRUIK

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BELL

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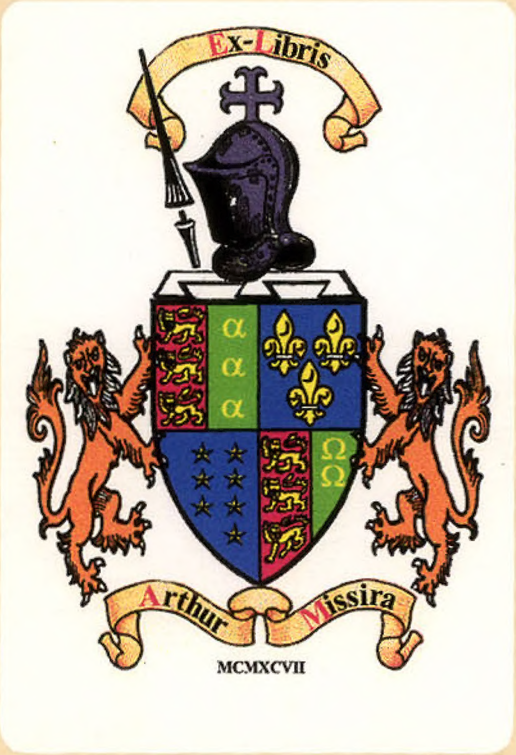
Professor Struik has achieved the seemingly impossible task of compressing the history of mathematics into less than three hundred pages. By stressing the unfolding of a few main ideas and by minimizing references to other developments, the author has been able to follow Egyptian, Babylonian, Chinese, Indian, Greek, Arabian, and Western mathematics from the earliest records to the beginning of the present century. He has based his account of nineteenth century advances on persons and schools rather than on subjects as the treatment by subjects has already been used in existing books. Important mathematicians whose work is analysed in detail are Euclid, Archimedes, Diophantos, Hammurabi, Bernoulli, Fermat, Euler, Newton, Leibniz, Laplace, Lagrange, Gauss, Jacobi, Riemann, Cremona, Betti, and others. Among the 47 illustrations are portraits of many of these great figures. Each chapter is followed by a select bibliography.



A CONCISE HISTORY

OF

MATHEMATICS

by

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LONDON
G. BELL AND SONS LTD
1954

Printed in Great Britain by
Butler & Tanner Ltd., Frome and London

FOR RUTH

CONTENTS

LIST OF ILLUSTRATIONS

1. Mathematics is a vast adventure in ideas; its history reflects some of the noblest thoughts of countless generations. It was possible to condense this history into a book of less than three hundred pages only by subjecting ourselves to strict discipline, sketching the unfolding of a few main ideas and minimizing reference to other developments. Bibliographical details had to be restricted to an outline; many relatively important authors—Roberval, Lambert, Schwarz—had to be bypassed. Perhaps the most crippling restriction was the insufficient reference to the general cultural and sociological atmosphere in which the mathematics of a period matured—or was stifled. Mathematics has been influenced by agriculture, commerce and manufacture, by warfare, engineering and philosophy, by physics and by astronomy. The influence of hydrodynamics on function theory, of Kantianism and of surveying on geometry, of electromagnetism on differential equations, of Cartesianism on mechanics and of scholasticism on the calculus could only be indicated in a few sentences—or perhaps a few words—yet an understanding of the course and content of mathematics can only be reached if all these determining factors are taken into consideration. Often a reference to the literature has had to replace an historical analysis. Our story ends by 1900, for we do not feel competent to judge the work of our contemporaries.

We hope that despite these restrictions we have been able to give a fairly honest description of the main

trends in the development of mathematics throughout the ages and of the social and cultural setting in which it took place. The selection of the material was, of course, not exclusively based on objective factors, but was influenced by the author's likes and dislikes, his knowledge and his ignorance. As to his ignorance, it was not always possible to consult all sources first-hand; too often second- or even third-hand sources had to be used. It is therefore good advice, not only with respect to this book, but with respect to all such histories, to check the statements as much as possible with the original sources. This is a good principle for more reasons than one. Our knowledge of authors such as Euclid, Diophantos, Descartes, Laplace, Gauss, or Riemann should not be obtained exclusively from quotations or histories describing their works. There is the same invigorating power in the original Euclid or Gauss as there is in the original Shakespeare, and there are places in Archimedes, in Fermat, or in Jacobi which are as beautiful as Horace or Emerson.

* * * * * *

Among the principles which have led the author in the presentation of his material are the following:

1. Stress the continuity and affinity of the Oriental civilizations, rather than the mechanical divisions between Egyptian, Babylonian, Chinese, Indian, and Arabian cultures.
2. Distinguish between established fact, hypothesis, and tradition, especially in Greek mathematics.
3. Relate the two trends in Renaissance mathematics, the arithmetical-algebraic and the "fluctional," respectively, to the commercial and the engineering interests of the period.
4. Base the exposition of Nineteenth Century mathematics on persons and schools rather than on subjects [Here Felix Klein's history could be used as a primary guide. An exposition by subjects can be found in the books by Cajori and Bell, or with the more technical details, in the "Encyklopædie der mathematischen Wissenschaften" (Leipzig, 1898-1935, 24 vols.) and in Pascal's "Repertorium der höheren Mathematik" (Leipzig 1910-29, 5 vols.).]

2. We list here some of the most important books on the history of mathematics as a whole. Such a list is superfluous for those who can consult G. Sarton, *The Study of the History of Mathematics* (Cambridge, 1936, 103 pp.) which not only has an interesting introduction to our subject but also has complete bibliographical information.

English texts to be consulted are:

R. C. Archibald, *Outline of the History of Mathematics* (Amer. Math. Monthly 561, Jan. 1949, 6th ed.)

This issue of 114 pp. contains an excellent summary and many bibliographic references.

F. Cajori, *A History of Mathematics* (London, 2nd ed., 1938).

This is a standard text of 514 pp.

D. E. Smith, *History of Mathematics* (London, 1923–25, 2 vols.).

This book is mainly restricted to elementary mathematics but

has references concerning all leading mathematicians. It contains many illustrations.

E. T. Bell, *Men of Mathematics* (Pelican Books, 1953).

E. T. Bell, *The Development of Mathematics* (New York-London, 2nd ed., 1945).

These two books contain a wealth of material, both on the mathematicians and on their works. The emphasis of the second book is on modern mathematics.

Dealing mainly with elementary mathematics are:

V. Sanford, *A Short History of Mathematics* (London, 1930).

W. W. Rouse Ball, *A Short Account of the History of Mathematics* (London, 6th. ed., 1915).

An older, very readable but antiquated text.

The standard work on the history of mathematics is still:

M. Cantor, *Vorlesungen über Geschichte der Mathematik* (Leipzig, 1900-08, 4 vols.).

This enormous work, of which the fourth volume was written by a group of specialists under Cantor's direction, covers the history of mathematics until 1799. It is here and there antiquated and often incorrect in details, but it remains a good book for a first orientation.

Corrections by G. Eneström a. o. in the volumes of "Bibliotheca mathematica."

Other German books are:

H. G. Zeuthen, *Geschichte der Mathematik im Altertum und Mittelalter* (Copenhagen, 1896; French ed., Paris, 1902).

H. G. Zeuthen, *Geschichte der Mathematik im XVI und XVII Jahrhundert* (Leipzig, 1903).

S. Günther—H. Wieleitner, *Geschichte der Mathematik* (Leipzig, 2 vols.). I (1908) written by Günther; II (1911-21, 2 parts) written by Wieleitner. Ed. by Wieleitner (Berlin, 1939).

J. Tropfke, *Geschichte der Elementar-Mathematik* (Leipzig, 2d ed., 1921-24, 7 vols.; vols. I-IV in 3d ed. 1930-34).

Die Kultur der Gegenwart III, 1 (Leipzig-Berlin, 1912); contains:

H. G. Zeuthen, *Die Mathematik im Altertum und im Mittelalter;*
A. Voss, *Die Beziehungen der Mathematik zur allgemeinen Kultur;*
H. E. Timerding, *Die Verbreitung mathematischen Wissens und mathematischer Auffassung.*

The oldest text book in the history of mathematics is in French:

J. E. Montucla, *Histoire des mathématiques* (Paris, new ed., 1799-1802, 4 vols.).

This book, first published in 1758 (2 vols.), also deals with applied mathematics. It is still good reading.

A good Italian book is:

G. Loria, *Storia delle matematiche* (Turin, 1929-33, 3 vols.).

There also exist anthologies of mathematical works:

D. E. Smith, *A Source Book in Mathematics* (London, 1929).

H. Wieleitner, *Mathematische Quellenbücher* (Berlin, 1927-29, 4 small vols.).

A. Speiser, *Klassische Stücke der Mathematik* (Zürich-Leipzig, 1925).

There also exist histories of special subjects of which we must mention:

L. E. Dickson, *History of the Theory of Numbers* (Washington, 1919-27, 3 vols.).

T. Muir, *The Theory of Determinants in the Historical Order of Development* (London, 1906-23, 4 vols.); with supplement, *Contributions to the History of Determinants 1900-20* (London, 1930).

A. von Braunmühl, *Vorlesungen über Geschichte der Trigonometrie* (Leipzig, 1900-03, 2 vols.).

T. Dantzig, *Number. The Language of Science* (New York, 3rd ed., 1943, also London, 1940).

J. L. Coolidge, *A History of Geometrical Methods* (Oxford, 1940).

G. Loria, *Il passato e il presente delle principali teorie geometriche* (Turin, 4th. ed., 1931).

G. Loria, *Storia della geometria descrittiva dalle origini sino ai giorni nostri* (Milan, 1921).

G. Loria, *Curve piani speciali algebriche e transcendenti* (Milan, 1930, 2 vols.); German edition (previously published Leipzig, 1910-11, 2 vols.).

F. Cajori, *A History of Mathematical Notations* (Chicago, 1928-29, 2 vols.).

L. C. Karpinski, *The History of Arithmetic* (Chicago, 1925).

H. M. Walker, *Studies in the History of Statistical Methods* (Baltimore, 1929).

R. Reiff, *Geschichte der unendlichen Reihen* (Tübingen, 1889).

I. Todhunter, *History of the Progress of the Calculus of Variations during the Nineteenth Century* (Cambridge, 1861).

I. Todhunter, *History of the Mathematical Theory of Probability from the Time of Pascal to that of Laplace* (Cambridge, 1865).

I. Todhunter, *A History of the Mathematical Theories of Attraction and the Figure of the Earth from the Time of Newton to that of Laplace* (London, 1873).

J. L. Coolidge, *The Mathematics of Great Amateurs* (Oxford, 1949).

R. C. Archibald, *Mathematical Table Makers* (New York, 1948).

Other books will be mentioned at the end of the different chapters.

The history of mathematics is also discussed in the books on the history of science in general. The standard work is:

G. Sarton, *Introduction to the History of Science* (Washington–Baltimore, 1927–48, 3 vols.).

This leads up to the Fourteenth Century[1] and can be supplemented by the essay:

G. Sarton, *The Study of the History of Science, with an Introductory Bibliography* (Cambridge, 1936).[2]

A good text for school use is:

W. T. Sedgwick—H. W. Tyler, *A Short History of Science* (New York, 3rd ed., 1948).

Also useful are the ten articles by G. A. Miller called *A first lesson in the history of mathematics*, *A second lesson*, etc., in "National Mathematics Magazine" Vols. 13 (1939)-19 (1945).

Periodicals dealing with the history of mathematics or of science in general are:

"Bibliotheca mathematica," ser. 1-3 (1884-1914)
"Scripta mathematica" (1932-present)
"Isis" (1913-present)

The author wishes to express his appreciation to Dr. Neugebauer whose willingness to read the first chapters of A CONCISE HISTORY OF MATHEMATICS has resulted in several improvements.

[1]We have followed in this book Sarton's transcription of Greek and Oriental names.

[2]See also G. Sarton's book mentioned on p. 286.

In this reprint we have corrected several misprints and errors which had slipped into the first printing. We would like to express to R. C. Archibald, E. J. Dÿksterhuis, S. A. Joffe, and other readers our appreciation of their help in detecting these inaccuracies.

CHAPTER I

The Beginnings

1. Our first conceptions of number and form date back to times as far removed as the Old Stone Age, the Paleolithicum. Throughout the hundreds or more millennia of this period men lived in caves, under conditions differing little from those of animals, and their main energies were directed towards the elementary process of collecting food wherever they could get it. They made weapons for hunting and fishing, developed a language to communicate with each other, and in the later paleolithic ages enriched their lives with creative art forms, statuettes and paintings. The paintings in caves of France and Spain (perhaps c. 15000 years ago) may have had some ritual significance; certainly they reveal a remarkable understanding of form.

Little progress was made in understanding numerical values and space relations until the transition occurred from the mere *gathering* of food to its actual *production*, from hunting and fishing to agriculture. With this fundamental change, a revolution in which the passive attitude of man toward nature turned into an active one, we enter the New Stone Age, the Neolithicum.

This great event in the history of mankind occurred perhaps ten thousand years ago, when the ice sheet which covered Europe and Asia began to melt and made room for forests and deserts. Nomadic wandering in search of food came slowly to an end. Fishermen and hunters were in large part replaced by primitive farmers.

Such farmers, remaining in one place as long as the soil stayed fertile, began to build more permanent dwellings; villages emerged as protection against the climate and against predatory enemies. Many such neolithic settlements have been excavated. The remains show how gradually elementary crafts such as pottery, carpentry, and weaving developed. There were granaries, so that the inhabitants were able to provide against winter and hard times by establishing a surplus. Bread was baked, beer was brewed, and in late neolithic times copper and bronze wère smelted and prepared. Inventions were made, notably of the potter's wheel and the wagon wheel; boats and shelters were improved. All these remarkable innovations occurred only within local areas and did not always spread to other localities. The American Indian, for example, did not learn of the existence of the wagon wheel until the coming of the white man. Nevertheless, as compared with the paleolithic times, the tempo of technical improvement was enormously accelerated.

Between the villages a considerable trade existed, which so expanded that connections can be traced between places hundreds of miles apart. The discovery of the arts of smelting and manufacturing, first copper then bronze tools and weapons, strongly stimulated this commercial activity. This again promoted the further formation of languages. The words of these languages expressed very concrete things and very few abstractions, but there was already some room for simple numerical terms and for some form relations. Many Australian, American, and African tribes were in this stage at the period of their first contact with white

men; some tribes are still living in these conditions so that it is possible to study their habits and forms of expression.

2. Numerical terms—expressing some of "the most abstract ideas which the human mind is capable of forming," as Adam Smith has said—came only slowly into use. Their first occurrence was qualitative rather than quantitative, making a distinction only between one (or better "a"—"a man"—rather than "one man") and two and many. The ancient qualitative origin of numerical conceptions can still be detected in the special dual terms existing in certain languages such as Greek or Celtic. When the number concept was extended higher numbers were first formed by addition: 3 by adding 2 and 1, 4 by adding 2 and 2, 5 by adding 2 and 3.

Here is an example from some Australian tribes:

Murray River: 1 = enea, 2 = petcheval, 3 = petcheval-enea, 4 = petcheval petcheval.
Kamilaroi: 1 = mal, 2 = bulan, 3 = guliba, 4 = bulan bulan, 5 = bula guliba, 6 = guliba guliba[1].

The development of the crafts and of commerce stimulated this crystallization of the number concept. Numbers were arranged and bundled into larger units, usually by the use of the fingers of the hand or of both hands, a natural procedure in trading. This led to numeration first with five, later with ten as a base, completed by addition and sometimes by subtraction,

[1]L. Conant, *The Number Concept* (London, 1896), pp. 106–107, with many similar examples.

so that twelve was conceived as $10 + 2$, or 9 as $10 - 1$. Sometimes 20, the number of fingers and toes, was selected as a base. Of 307 number systems of primitive American peoples investigated by W. C. Eels, 146 were decimal, 106 quinary and quinary decimal, vigesimal and quinary vigesimal[1]. The vigesimal system in its most characteristic form occurred among the Mayas of Mexico and the Celts in Europe.

Numerical records were kept by means of bundling, strokes on a stick, knots on a string, pebbles or shells arranged in heaps of fives—devices very much like those of the old time inn-keeper with his tally stick. From this method to the introduction of special symbols for 5, 10, 20, etc. was only a step, and we find exactly such symbols in use at the beginning of written history, at the so-called dawn of civilization.

The oldest example of the use of a tally stick dates back to paleolithic times and was found in 1937 in Vestonice (Moravia). It is the radius of a young wolf, 7 in. long, engraved with 55 deeply encised notches, of which the first 25 are arranged in groups of 5. They are followed by a simple notch twice as long which terminates the series; then, starting from the next notch, also twice as long, a new series runs up to 30[2].

It is therefore clear that the old saying found in Jacob Grimm and often repeated, that counting started as finger counting, is incorrect. Counting by fingers, that is, counting by fives and tens, came only at a certain stage of social development. Once it was reached, num-

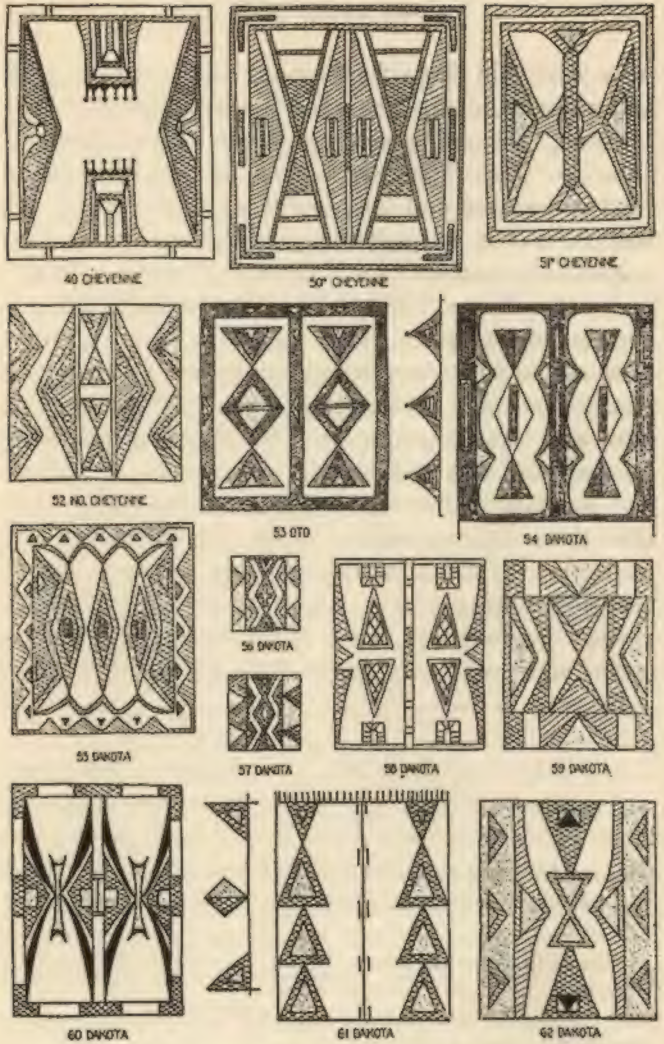
[1]W. C. Eels., *Number Systems of North American Indians*, Am. Math. Monthly 20 (1913), p. 293.

[2]Isis 28 (1938) pp. 462-463, from illustrated London News, Oct. 2, 1937.

bers could be expressed with reference to a base, with the aid of which large numbers could be formed; thus originated a primitive type of arithmetic. Fourteen was expressed as $10 + 4$, sometimes as $15 - 1$. Multiplication began where 20 was expressed not as $10 + 10$, but as 2×10. Such dyadic operations were used for millennia as a kind of middle road between addition and multiplication, notably in Egypt and in the pre-Aryan civilization of Mohenjo-Daro on the Indus. Divisions began where 10 was expressed as "half of a body", though conscious formation of fractions remained extremely rare. Among North American tribes, for instance, only a few instances of such formations are known, and this is in almost all cases only of 1/2, although sometimes also of 1/3 or 1/4.[1] A curious phenomenon was the love of very large numbers, a love perhaps stimulated by the all-too-human desire to exaggerate the extent of herds or of enemies slain; remnants of this tendency appear in the Bible and in other sacred writings.

3. It also became necessary to measure the length and contents of objects. The standards were rough and often taken from parts of the human body, and in this way units originated like fingers, feet, or hands. Names like ell, fathom, cubit also remind us of this custom. When houses were built, as among the agricultural Indians or the pole house dwellers of Central Europe,

[1]G. A. Miller has remarked that the words *one-half*, *semis*, *moitié* have no direct connection with the words *two*, *duo*, *deux* (contrary to *one-third*, *one-fourth*, etc.), which seems to show that the conception of ½ originated independent of that of integer. Nat. Math. Magazine 13 (1939) p. 272.



GEOMETRICAL PATTERNS DEVELOPED BY AMERICAN INDIANS
(From Spier)

rules were laid down for building along straight lines and at right angles. The word "straight" is related to "stretch," indicating operations with a rope[1]; the work "line" to "linen," showing the connection between the craft of weaving and the beginnings of geometry. This was one way in which interest in mensuration evolved.

Neolithic man also developed a keen feeling for geometrical patterns. The baking and coloring of pottery, the plaiting of rushes, the weaving of baskets and textiles, and later the working of metals led to the cultivation of plane and spatial relationships. Dance patterns must also have played a role. Neolithic ornamentation rejoiced in the revelation of congruence, symmetry, and similarity. Numerical relationships might enter into these figures, as in certain prehistoric patterns which represent triangular numbers; others display "sacred" numbers.

Here follow some interesting geometrical patterns occurring in pottery, weaving or basketry.

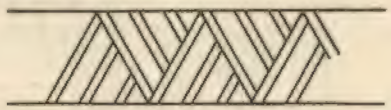


Fig. 1.

This can be found on neolithic pottery in Bosnia and on objects of art in the Mesopotamian Ur-period[2].

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[1]The name "rope-stretchers" (Greek: "harpedonaptai," Arabic: "massah," Assyrian: "masiḫānu") was attached in many countries to men engaged in surveying—see S. Gandz, *Quellen und Studien zur Geschichte der Mathematik* I (1930) pp. 255-277.

[2]W. Lietzmann, *Geometrie und Praehistorie*, Isis 20(1933) pp. 436-439.

FIG. 2.

This exists on Egyptian pottery of the Predynastic period (4000-3500 B.C.)[1].

FIG. 3.

These patterns were used by pole house dwellers near Ljubljana (Yugoslavia) in the Hallstatt-period (Central Europe, 1000-500 B.C.)[2].

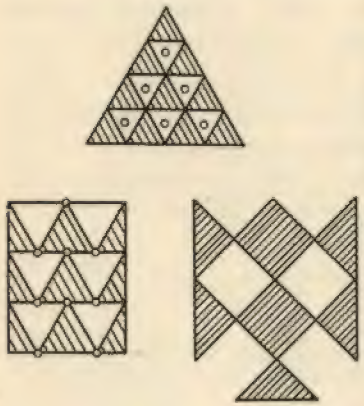

FIG. 4.

These rectangles filled with triangles, triangles filled with circles, are from urns in graves near Sopron in Hungary. They show attempts at the formation of triangular numbers, which played an important role in Pythagorean mathematics of a later period.[3]

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[1]D. E. Smith, *History of Mathematics* (Ginn & Co., 1923) I p 15.

[2]M. Hoernes, *Urgeschichte der bildenden Kunst in Europa* (Vienna, 1915).

[3]See also F. Boas, *General Anthropology* (1930) p. 273.

Patterns of this kind have remained popular throughout historical times. Beautiful examples can be found on dipylon vases of the Minoan and early Greek periods, in the later Byzantine and Arabian mosaics, on Persian and Chinese tapestry. Originally there may have been a religious or magic meaning to the early patterns, but their esthetic appeal gradually became dominant.

In the religion of the Stone Age we can discern a primitive attempt to contend with the forces of nature. Religious ceremonies were deeply permeated with magic, and this magical element was incorporated into existing conceptions of number and form as well as in sculpture, music, and drawing. There were magical numbers, such as 3, 4, 7 and magical figures such as the Pentalpha and the Swastica. Some authors have even considered this aspect of mathematics the determining factor in its growth[1], but though the social roots of mathematics may have become obscured in modern times, they are fairly obvious during the early section of man's history. "Modern" numerology is a leftover from magical rites dating back to neolithic, and perhaps even to paleolithic, times.

4. Even among very primitive tribes we find some reckoning of time and, consequently, some knowledge of the motion of sun, moon, and stars. This knowledge attained its first more scientific character when farming and trade expanded. The use of a lunar calendar goes very far back into the history of mankind, the changing aspects of vegetation being connected with the changes

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[1]W. J. McGee, *Primitive Numbers*, Nineteenth Annual Report, Bureau Amer. Ethnology 1897-98 (1900) pp. 825-851.

Courtesy of The Metropolitan Museum of Art

PREDYNASTIC EGYPTIAN POTTERY

of the moon. Primitive people also pay attention to the solstices or rising of the Pleiades at dawn. The earliest civilized people attributed a knowledge of astronomy to their most remote, prehistoric periods. Other primitive peoples used the constellations as guides in navigation. From this astronomy resulted some knowledge of the properties of the sphere, of angular directions, and of circles.

5. These few illustrations of the beginnings of mathematics show that the historical growth of a science does not necessarily pass through the stages in which we now develop it in our instruction. Some of the oldest geometrical forms known to mankind, such as knots and patterns, only received full scientific attention in recent years. On the other hand some of our more elementary branches of mathematics, such as the graphical representation or elementary statistics, date back to comparatively modern times. As A. Speiser has remarked with some asperity: "Already the pronounced tendency toward tediousness, which seems to be inherent in elementary mathematics, might plead for its late origin, since the creative mathematician would prefer to pay his attention to the interesting and beautiful problems."[1]

Literature.

Apart from the texts by Conant, Eels, Smith, Lietzmann, McGee, and Speiser already quoted, see:

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[1]A. Speiser, *Theorie der Gruppen von endlicher Ordnung* (Leipzig 1925, reprint New York 1945) p. 3.

R. Menninger, *Zahlwort und Ziffer* (Breslau, 1934).

D. E. Smith–J. Ginsburg, *Numbers and Numerals* (N. Y. Teachers' College, 1937).

Gordon Childe, *What Happened in History* (Pelican Book, 1942).

Interesting patterns are described in:

L. Spier, *Plains Indian Parfleche Designs*, Un. of Washington Publ. in Anthropology 4 (1931), pp. 293–322.

A. B. Deacon, *Geometrical drawings from Malekula and other Islands of the New Hebrides*, Journ. Roy. Anthrop. Institute 64 (1934) pp. 129-175.

M. Popova, *La géométrie dans la broderie bulgare*, Comptes Rendus, Premier Congrès des Mathématiciens des pays slaves (Warsaw, 1929) pp. 367-369.

On the mathematics of the American Indians see also:

J. E. S. Thompson, *Maya Arithmetic*, Contributions to Am. Anthropology and History 36, Carnegie Inst. of Washington Publ. 528 (1941) pp. 37-62.

An extensive bibliography in D. E. Smith, *History of Mathematics* I (1923) p. 14.

For a bibliography on the development of mathematical concepts in children:
A. Riess, *Number Readiness in Research* (Chicago, 1947).

CHAPTER II

The Ancient Orient

1. During the fifth, fourth and third millennium B.C. newer and more advanced forms of society evolved from well established neolithic communities along the banks of great rivers in Africa and Asia, in sub-tropic or nearly sub-tropic regions. These rivers were the Nile, the Tigris and Euphrates, the Indus and later the Ganges, the Hoang-ho and later the Yang-tse.

The lands along these rivers could be made to grow abundant crops once the flood waters were brought under control and the swamps drained. By contrast to the arid desert and mountain regions and plains surrounding these countries the river valleys could be made into a paradise. Within the course of the centuries these problems were solved by the building of levees and dams, the digging of canals, and the construction of reservoirs. Regulation of the water supply required coordination of activities between widely separated localities on a scale greatly surpassing all previous efforts. This led to the establishment of central organs of administration, located in urban centers rather than in the barbarian villages of former periods. The relatively large surplus yielded by the vastly improved and intensive agriculture raised the standard of living for the population as a whole, but it also created an urban aristocracy headed by powerful chieftains. There were many specialized crafts carried on by artisans, soldiers, clerks, and priests. Administration of the public works

3

was placed in the hands of a permanent officialdom, a group wise in the behavior of the seasons, the motions of the heavenly bodies, the art of land division, the storage of food, and the raising of taxes. A script was used to codify the requirements of the administration and the actions of the chieftains. Bureaucrats as well as artisans acquired a considerable amount of technical knowledge, including metallurgy and medicine. To this knowledge belonged also the arts of computation and mensuration.

By now social classes were firmly established. There were chieftains, free and tenant farmers, craftsmen, scribes, and officials, serfs and slaves. Local chiefs so increased in wealth and power that they rose from feudal lords of limited authority to become local kings of absolute sovereignty. Quarrels and wars among the various despots led to larger domains, united under a single monarch. These forms of society based on irrigation and intensive agriculture led in this way to an "Oriental" form of despotism. Such despotism could be maintained for centuries and then collapse, sometimes under the impact of mountain and desert tribes attracted by the wealth of the valleys, or again through neglect of the vast, complicated, and vital irrigation system. Under such circumstances power might shift from one tribal king to another, or society might break up into smaller feudal units, and the process of unification would start all over again. However, under all these dynastic revolutions and recurrent transitions from feudalism to absolutism the villages, which were the basis of this society, remained essentially unchanged, and with it the fundamental economic and

social structure. Oriental society moves in cycles, and there exist even at present many communities in Asia and Africa which have persisted for several millennia in the same pattern of life. Progress under such conditions was slow and erratic, and periods of cultural growth might be separated by many centuries of stagnation and decay.

The static character of the Orient imparted a fundamental sanctity to its institutions which facilitated the identification of religion with the state apparatus. The bureaucracy often shared this religious character of the state; in many oriental countries priests were the administrators of the domain. Since the cultivation of science was the task of the bureaucracy, we find in many—but not all—oriental countries that the priests were the outstanding carriers of scientific knowledge.

2. Oriental mathematics originated as a practical science to facilitate computation of the calendar, administration of the harvest, organization of public works, and collection of taxes. The initial emphasis was naturally on practical arithmetic and mensuration. However, a science cultivated for centuries by a special craft, whose task it is not only to apply it but also to instruct its secrets, develops tendencies toward abstraction. Gradually it will come to be studied for its own sake. Arithmetic evolved into algebra not only because it allowed better practical computations, but also as the natural outgrowth of a science cultivated and developed in schools of scribes. For the same reasons mensuration developed into the beginnings—but no more—of a theoretical geometry.

Despite all the trade and commerce in which these ancient societies indulged, their economic core was agricultural, centered in the villages, characterized by isolation and traditionalism. The result was that despite similarity in economic structure and in the essentials of scientific lore, there always remained striking differences between the different cultures. The seclusion of the Chinese and of the Egyptians was proverbial. It always has been easy to differentiate between the arts and the script of the Egyptians, the Mesopotamians, the Chinese, and the Indians. We can in the same way speak of Egyptian, Mesopotamian, Chinese, and Indian mathematics, though their general arithmetic-algebraic nature was very much alike. Even if the science of one country progressed beyond that of another during some period, it preserved its characteristic approach and symbolism.

It is difficult to date new discoveries in the East. The static character of its social structure tends to preserve scientific lore throughout centuries or even millennia. Discoveries made within the seclusion of a township may never spread to other localities. Storages of scientific and technical knowledge can be destroyed by dynastic changes, wars, or floods. The story goes that in 221 B.C. when China was united under one absolute despot, Shih Huang Ti (the Great Yellow Emperor), he ordered all books of learning to be destroyed. Later much was rewritten by memory, but such events make the dating of discoveries very difficult.

Another difficulty in dating Oriental science is due

to the material used for its preservation. The Mesopotamian people baked clay tablets which are virtually indestructible. The Egyptians used papyrus and a sizeable body of their writing has been preserved in the dry climate. The Chinese and Indians used far more perishable material, such as bark or bamboo. The Chinese, in the Second Century A.D., began to use paper, but little has been preserved which dates back to the millennia before 700 A.D. Our knowledge of Oriental mathematics is therefore very sketchy; for the pre-Hellenistic centuries we are almost exclusively confined to Mesopotamian and Egyptian material. It is entirely possible that new discoveries will lead to a complete re-evaluation of the relative merits of the different Oriental forms of mathematics. For a long time our richest historical field lay in Egypt because of the discovery in 1858 of the so-called Papyrus Rhind, written about 1650 B.C., but which contained much older material. In the last twenty years our knowledge of Babylonian mathematics has been vastly augmented by the remarkable discoveries of O. Neugebauer and F. Thureau-Dangin, who decyphered a large number of clay tablets. It has now appeared that Babylonian mathematics was far more developed than its Oriental counterparts. This judgement may be final, since there exists a certain consistency in the factual character of the Babylonian and Egyptian texts throughout the centuries. Moreover, the economic development of Mesopotamia was more advanced than that of other countries in the so-called Fertile Crescent of the Near East, which stretched from Mesopotamia to Egypt. Meso-

potamia was the crossroads for a large number of caravan routes, while Egypt stood in comparative isolation. Added to this was the fact that the harnessing of the erratic Tigris and Euphrates required more engineering skill and administration than that of the Nile, that "most gentlemanly of all rivers," to quote Sir William Willcocks. Further study of ancient Hindu mathematics may still reveal unexpected excellence, though so far claims for it have not been very convincing.

3. Most of our knowledge of Egyptian mathematics is derived from two mathematical papyri: one the Papyrus Rhind, already mentioned and containing 85 problems; the other the so-called Moscow Papyrus, perhaps two centuries older, containing 25 problems. These problems were already ancient lore when the manuscripts were compiled, but there are minor papyri of much more recent date—even from Roman times—which show no difference in approach. The mathematics they profess is based on a decimal system of numeration with special signs for each higher decimal unit—a system with which we are familiar through the Roman system which follows the same principle: MDCCCLXXVIII = 1878. On the basis of this system the Egyptians developed an arithmetic of a predominantly additive character, which means that its main tendency was to reduce all multiplication to repeated additions. Multiplication by 13, for instance, was obtained by multiplying first by 2, then by 4, then by 8, and adding the results of multiplication by 4 and 8 to the original number.

E.g. for the computation of 13×11:

*1	11
2	22
*4	44
*8	88

Add the numbers indicated by *, which gives 143.

Many problems were very simple and did not go beyond a linear equation with one unknown:

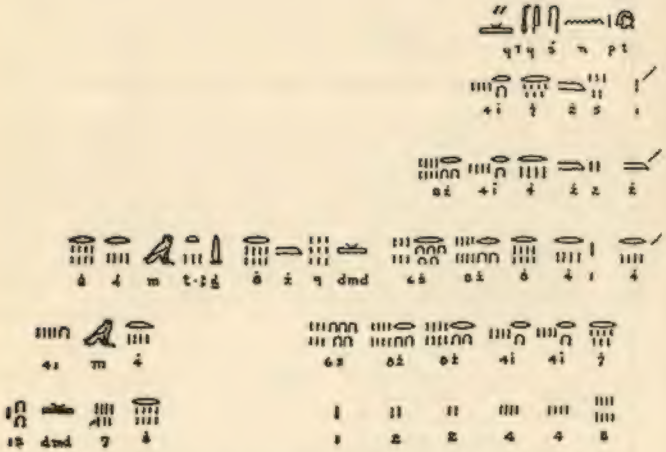
A quantity, its 2/3, its 1/2, and its 3/7, added together becomes 33. What is the quantity?

The most remarkable aspect of Egyptian arithmetic was its calculus of fractions. All fractions were reduced to sums of so-called unit fractions, meaning fractions with one as numerator. The only exception was $\frac{2}{3} = 1 - \frac{1}{3}$, for which there was a special symbol in existence. The reduction to sums of unit fractions was made possible by tables, which gave the decomposition for fractions of the form $2/n$—the only decomposition necessary because of the dyadic multiplication. The Papyrus Rhind has a table giving the equivalents in unit fractions for all odd n from 5 to 331, e.g.

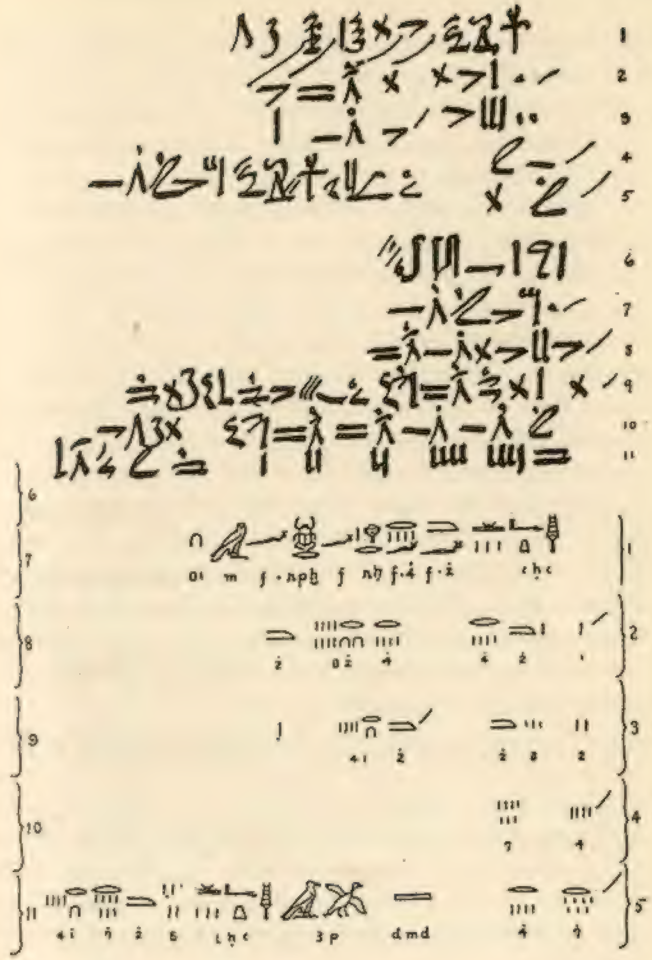
$$\frac{2}{7} = \frac{1}{4} + \frac{1}{28}$$

$$\frac{2}{97} = \frac{1}{56} + \frac{1}{679} + \frac{1}{776}$$

Such a calculus with fractions gave to Egyptian mathematics an elaborate and ponderous character, and



A SINGLE PAGE OF THE GREAT PAPYRUS RHIND
(From Chase, II, plate 56)



effectively impeded the further growth of science. This decomposition presupposed at the same time some mathematical skill, and there exist interesting theories to explain the way in which the Egyptian specialists might have obtained their results.[1]

The problems deal with the strength of bread and of different kinds of beer, with the feeding of animals and the storage of grain, showing the practical origin of this cumbersome arithmetic and primitive algebra. Some problems show a theoretical interest, as in the problem of dividing 100 loaves among 5 men in such a way that the share received shall be in arithmetical progression, and that one seventh of the sum of the largest three shares shall be equal to the sum of the smallest two. We even find a geometrical progression dealing with 7 houses in each of which there are 7 cats, each cat watching 7 mice, etc., which reveals a knowledge of the formula of the sum of a geometrical progression.

Some problems were of a geometrical nature, dealing mostly with mensuration. The area of the triangle was found as half the product of base and altitude; the area of a circle with diameter d was given as $\left(d - \frac{d}{9}\right)^2$, which led to a value of π of $256/81 = 3.1605$. We also meet some formulas for solid volumes, such as the cube, the parallelepiped and the circular cylinder, all conceived concretely as containers, mainly of grain. The most remarkable result of Egyptian mensuration was

[1]O. Neugebauer, *Arithmetik und Rechentechnik der Ägypter*, Quellen und Studien zur Geschichte der Mathematik BI (1931) pp. 301-380. B. L. van der Waerden, *Die Entstehungsgeschichte der ägyptischen Bruchrechnung*, vol. 4 (1938) pp. 359-382.

the formula for the value of the frustrum of a square pyramid $V = (h/3)\,(a^2 + ab + b^2)$, where a and b are the lengths of the sides of the squares and h is the height. This result, of which no counterpart has so far been found in other ancient forms of mathematics, is the more remarkable since there is no indication that the Egyptians had any notion even of the Pythagorean theorem, despite some unfounded stories about "harpedonaptai", who supposedly constructed right triangles with the aid of a string with $3 + 4 + 5 = 12$ knots[1].

We must here warn against exaggerations concerning the antiquity of Egyptian mathematical knowledge. All kinds of advanced science have been credited to the pyramid builders of 3000 B.C. and before, and there is even a widely accepted story that the Egyptians of 4212 B.C. adopted the so-called Sothic cycle for the measurement of the calendar. Such precise mathematical and astronomical work cannot be seriously ascribed to a people slowly emerging from neolithic conditions, and the source of these tales can usually be traced to a late Egyptian tradition transmitted to us by the Greeks. It is a common characteristic of ancient civilizations to date fundamental knowledge back to very early times. All available texts point to an Egyptian mathematics of rather primitive standards. Their astronomy was on the same general level.

4. With Mesopotamian mathematics we rise to a far higher level than Egyptian mathematics ever obtained. We can here even detect progress in the course of the

[1]See S. Gandz, loc. cit. p. 7.

centuries. Already the oldest texts, dating from the latest Sumerian period (the Third Dynasty of Ur, c. 2100 B.C.), show keen computational ability. These texts contain multiplication tables in which a well developed sexagesimal system of numeration was superimposed on an original decimal system; there are cuneiform symbols indicating 1, 60, 3600, and also 60^{-1}, 60^{-2}. However, this was not their most characteristic feature. Where the Egyptians indicated each higher unit by a new symbol, these Sumerians used the same symbol but indicated its value by its *position*. Thus 1 followed by another 1 meant 61, and 5 followed by 6 followed by 3 (we shall write 5,6,3) meant $5 \times 60^2 + 6 \times 60 + 3 = 18363$. This position, or place value, system did not differ essentially from our own system of writing numbers, in which the symbol 343 stands for $3 \times 10^2 + 4 \times 10 + 3$. Such a system had enormous advantages for computation, as we can readily see when we try to perform a multiplication in our own system and in a system with Roman numerals. The position system also removed many of the difficulties of fractional arithmetic, just as our own decimal system of writing fractions does. This whole system seems to have been a direct result of the technique of administration, as is witnessed by thousands of texts dating from the same period and dealing with the delivery of cattle, grain, etc., and with arithmetical work based on these transactions.

In this type of reckoning existed some ambiguities since the exact meaning of each symbol was not always clear from its position. Thus (5,6,3) might also mean $5 \times 60^1 + 6 \times 60^0 + 3 \times 60^{-1} = 306\ 1/20$, and the

exact interpretation had to be gathered from the context. Another uncertainty was introduced through the fact that a blank space sometimes meant zero, so that (11,5) might stand for $11 \times 60^2 + 5 = 39605$. Eventually a special symbol for zero appeared, but not before the Persian era. The so-called "invention of the zero" was, therefore, a logical result of the introduction of the position system, but only after the technique of computation had reached a considerable perfection.

Both the sexagesimal system and the place value system remained the permanent possession of mankind. Our present division of the hour into 60 minutes and 3600 seconds dates back to these Sumerians, as well as our division of the circle into 360 degrees, each degree into 60 minutes and each minute into 60 seconds. There is reason to believe that this choice of 60 rather than 10 as a unit occurred in an attempt to unify systems of measure, though the fact that 60 has many divisors may also have played a role. As to the place value system, the permanent importance of which has been compared to that of the alphabet[1]—both inventions replacing a complex symbolism by a method easily understood by a large number of people—, its history is still wrapt in considerable obscurity. It is reasonable to suppose that both Hindus and Greeks made its acquaintance on the caravan routes through Babylon; we also know that the Arabs described it as an Indian invention. The Babylonian tradition, however, may have influenced all later acceptance of the position system.

[1]O. Neugebauer, *The History of Ancient Astronomy*, Journal of Near Eastern Studies 4 (1945) p. 12.

5. The next group of cuneiform texts dates back to the first Babylonian Dynasty, when King Hammurabi reigned in Babylon (1950 B.C.), and a semitic population had subdued the original Sumerians. In these texts we find arithmetic evolved into a well established algebra. Although the Egyptians of this period were only able to solve simple linear equations, the Babylonians of Hammurabi's days were in full possession of the technique of handling quadratic equations. They solved linear and quadratic equations in two variables, and even problems involving cubic and biquadratic equations. They formulated such problems only with specific numerical values for the coefficients, but their method leaves no doubt that they knew the general rule.

Here is an example taken from a tablet dating from this period:

"An area A, consisting of the sum of two squares is 1000. The side of one square is 2/3 of the side of the other square, diminished by 10. What are the sides of the square?"

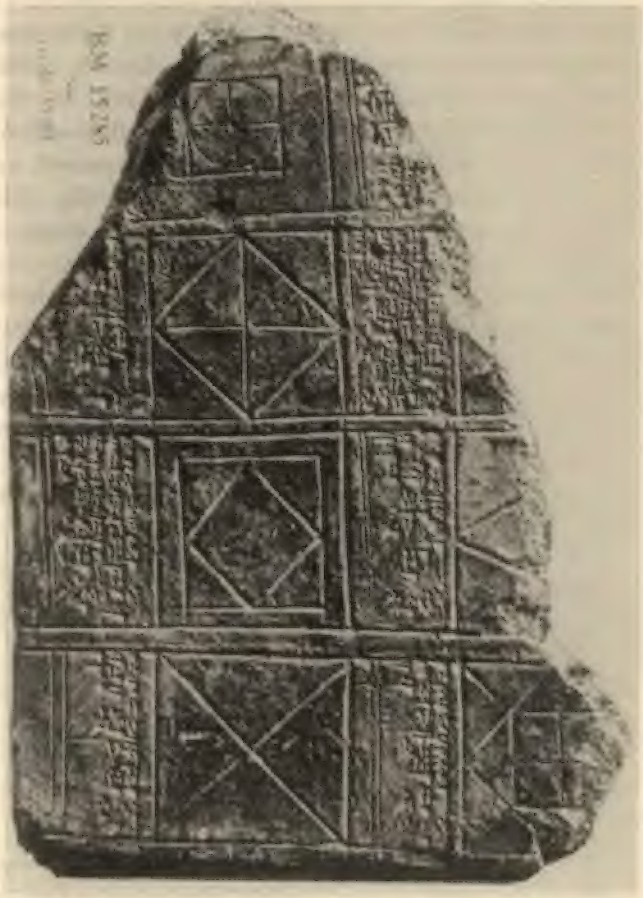
This leads to the equations $x^2 + y^2 = 1000$, $y = 2/3\,x - 10$ of which the solution can be found by solving the quadratic equation

$$\frac{13}{9}x^2 - \frac{40}{3}x - 900 = 0$$

which has one positive solution $x = 30$.

The actual solution in the cuneiform text confines itself—as in all Oriental problems—to the simple enumeration of the numerical steps that must be taken to solve the quadratic equation:

"Square 10: this gives 100; subtract 100 from 1000; this gives 900," etc.



ONE SIDE OF A CUNEIFORM TEXT NOW IN THE BRITISH MUSEUM
(From Neugebauer, Math. Keilschr. Texte, 3, II, plate 3)

The strong arithmetical-algebraic character of this Babylonian mathematics is also apparent from its geometry. As in Egypt, geometry developed from a foundation of practical problems dealing with mensuration, but the geometrical form of the problem was usually only a way of presenting an algebraic question. The previous example shows how a problem concerning the area of a square led to a non-trivial algebraic problem, and this example is no exception. The texts show that the Babylonian geometry of the Semitic period was in possession of formulas for the areas of simple rectilinear figures and for the volumes of simple solids, though the volume of a truncated pyramid has not yet been found. The so-called theorem of Pythagoras was known, not only for special cases, but in full generality. The main characteristic of this geometry was, however, its algebraic character. This is equally true of all later texts, especially those dating back to the third period of which we have a generous number of texts, that of the New Babylonian, Persian, and Seleucid eras (from c. 600 B.C.—300 A.D.).

The texts of this later period are strongly influenced by the development of Babylonian astronomy, which in those days assumed really scientific traits, characterized by a careful analysis of the different ephemerides. Mathematics became even more perfect in its computational technique; its algebra tackled problems in equations which even now require considerable numerical skill. There exist computations dating from the Seleucid period which go to seventeen sexagesimal places. Such complicated numerical work was no longer related to problems of taxation or mensuration, but was stimulated by astronomical problems or by pure love of computation.

Much of this computational arithmetic was done with tables, which ranged from simple multiplication tables to lists of reciprocals and of square and cubic roots. One table gives a list of numbers of the form $n^3 + n^2$, which was used, it seems, to solve cubic equations such as $x^3 + x^2 = a$. There were some excellent approximations, $\sqrt{2}$ was indicated by $1\frac{5}{12}$ ($\sqrt{2} = 1.4142$, $1\frac{5}{12} = 1.4167$)[1], and $1/\sqrt{2} = .7071$ by $17/24 = .7083$. Square roots seem to have been found by formulas like these:

$$\sqrt{A} = \sqrt{a^2 + h} = a + h/2a = \frac{1}{2}\left(a + \frac{A}{a}\right).$$

It is a curious fact that in Babylonian mathematics no better approximation for π has so far been found than the Biblical $\pi = 3$, the area of a circle being taken as $\frac{1}{12}$ of the square of its circumference.

The equation $x^3 + x^2 = a$ appears in a problem which calls for the solution of simultaneous equations $xyz + xy = 1 + 1/6$, $y = 2/3\, x$, $z = 12x$ which leads to $(12x)^3 + (12x)^2 = 252$, or $12x = 6$ (from the table).

There also exist cuneiform texts with problems in compound interest, such as the question of how long it would take for a certain sum of money to double itself

[1]O. Neugebauer, *Exact Sciences in Antiquity*, Un. of Pennsylvania Bicentennial Conference, Studies in Civilization, Philadelphia 1941, pp. 13–29 (Copenhagen, 1951).

at 20 percent interest. This leads to the equation $\left(1\frac{1}{5}\right)^x = 2$, which is solved by first remarking that $3 < x < 4$ and then by linear interpolation; in our way of writing

$$4 - x = \frac{(1.2)^4 - 2}{(1.2)^4 - (1.2)^3},$$

leading to $x = 4$ years minus (2,33,20) months.

One of the specific reasons for the development of algebra around 2000 B.C. seems to have been the use of the old Sumerian script by the new Semitic rulers, the Babylonians. The ancient script was, like the hieroglyphics, a collection of ideograms, each sign denoting a single concept. The Semites used them for the phonetic rendition of their own language and also took over some signs in their old meaning. These signs now expressed concepts, but were pronounced in a different way. Such ideograms were well fitted for an algebraic language, as are our present signs $+$, $-$, :, etc., which are really also ideograms. In the schools for administrators in Babylon this algebraic language became a part of the curriculum for many generations, and though the empire passed through the hands of many rulers—Kassites, Assyrians, Medes, Persians—the tradition remained.

The more intricate problems date back to later periods in the history of ancient civilization, notably to the Persian and Seleucid times. Babylon, in those days, was no longer a political center, but remained for many centuries the cultural heart of a large empire, where Babylonians mixed with Persians, Greeks, Jews, Hindus, and many other peoples. There is in all the cuneiform texts a continuity of tradition which seems to point to a continuous local development. There is little doubt that this local development was also stimulated by the contact with other civilizations and that this stimulation acted both ways. We know that Babylonian astronomy of this period influenced Greek astronomy and that Babylonian mathematics influenced computational arithmetic; it is reasonable to assume that through the medium of the Babylonian schools of scribes, Greek science and Hindu science met. The role of Persian and Seleucid Mesopotamia in the spread of ancient and antique astronomy and mathematics is still poorly known, but all available evidence shows that it must have been considerable. Medieval Arabic and Hindu science did not only base itself on the tradition of Alexandria but also on that of Babylon.

6. Nowhere in all ancient Oriental mathematics do we find any attempt at what we call a demonstration. No argumentation was presented, but only the prescription of certain rules: "Do such, do so". We are ignorant of the way in which the theorems were found: how, for instance, did the Babylonians become acquainted with the theorem of Pythagoras? Several attempts exist to explain the way in which Egyptians and Babylonians obtained their results, but they are all of an hypothetical nature. To us who have been educated on Euclid's strict argumentation, this whole Oriental way of reasoning seems at first strange and highly unsatisfactory. But this strangeness wears off when we realize that most of

the mathematics we teach our present day engineers and technicians is still of the "Do such, do so" type, without much attempt at rigorous demonstration. Algebra is still being taught in many high schools as a set of rules rather than a science of deduction. Oriental mathematics never seems to have been emancipated from the millennial influence of the problems in technology and administration, for the use of which it had been invented.

7. The question of Greek and Babylonian influence determines profoundly the study of ancient Hindu and Chinese mathematics. The native Indian and Chinese scholars of later days used—and sometimes still use—to stress the great antiquity of their mathematics, but there are no mathematical texts in existence which can be definitely dated to the pre-Christian era. The oldest Hindu texts are perhaps from the first centuries A.D., the oldest Chinese texts date back to an even later period. We do know that the ancient Hindus used decimal systems of numeration without a place value notation. Such a system was formed by the so-called Brâhmî numerals, which had special signs for each of the numbers 1, 2, 3, ······· , 9, 10; 20, 30, 40, ····· , 100; 200, 300, ······· , 1000; 2000, ··· ; these symbols go back at least to the time of King Açoka (300 B.C.).

Then there exist the so-called "Súlvasūtras," of which parts date back to 500 B.C. or earlier, and which contain mathematical rules which may be of ancient native origin. These rules are found among ritualistic prescriptions, of which some deal with the construction of altars. We find here recipes for the construction of

squares and rectangles and expressions for the relation of the diagonal to the sides of the square and for the equivalence of circles and squares. There is some knowledge of the Pythagorean theorem in specific cases, and there are a few curious approximations in terms of unit fractions, such as (in our notation):

$$\sqrt{2} = 1 + \frac{1}{3} + \frac{1}{3.4} - \frac{1}{3.4.34} \text{ (= 1.4142156)}$$

$$\pi = 4\left(1 - \frac{1}{8} + \frac{1}{8.29} - \frac{1}{8.29.6} + \frac{1}{8.29.6.8}\right)^2$$

$$= 18(3 - 2\sqrt{2})$$

The curious fact that these results of the "Súlvasūtras" do not occur in later Hindu works shows that we cannot yet speak of that continuity of tradition in Hindu mathematics which is so typical of its Egyptian and of Babylonian counterparts, and this continuity may actually be absent, India being as large as it is. There may have been different traditions relating to various schools. We know, for instance, that Jainism, which is as ancient as Buddhism (c. 500 B.C.) encouraged mathematical studies; in Jaina sacred books the value $\pi = \sqrt{10}$ is found[1].

8. The study of ancient Chinese mathematics is considerably handicapped by the lack of satisfactory translations, so that we are forced to use second-hand sources,

[1]B. Datta, *The Jaina School of Mathematics*, Bulletin Calcutta Math. Soc. 21 (1929) pp. 115-146.

mainly the reports of Mikami, Biot and Biernatzki[1], which are all very sketchy. They give us some information about the so-called Ten Classics (the "suan-ching"), a collection of mathematical and astronomical texts used for the state examination of officials in the main mathematics section during the T'ang dynasty (618-907 A.D.) The material contained in these texts is much older; the first of them, the "Chou pi," is supposed to date back to the Chou period (1112-256 B.C.), and part of its content may have been of ancient date even at that period. One book outside of the Ten Classics, the "I-ching," is perhaps even more ancient than the "Chou pi"; it contains some mathematics among much divination and magic. Its best known mathematical contribution is the magic square

$$\begin{array}{ccc} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6. \end{array}$$

The number system of ancient China, as it appears from the Ten Classics, was decimal with special symbols for the higher units, like the Egyptian system. It seems that in order to express higher units the tokens for the lower units were repeated, so that a position system arose. We find, for instance, in the "Sun Tzŭ," one of the Ten Classics dating back to the first century A.D., a description of the use of calculating pieces for the performance of multiplication and division. These pieces were made of bamboos or wood and were arranged

[1]K. L. Biernatzki, *Die Arithmetik der Chinesen*, Crelle 52 (1856) pp. 59-94.

so that, for instance, |, ||, T, TT, —, =, ⊥ meant 1, 2, 6, 7, 10, 20, 60, respectively. This led to a position system with 20 tokens, in which ⊥ TT = TTTT meant 6729. Here the ingenious idea of a position system may have resulted from the difficulty of finding an unlimited number of single horizontal and vertical positions of the sticks.

In the computation of the calendar a kind of sexagesimal decimal system was used in which 60 was a higher unit called "cycle" (the "cycle in Cathay" of Tennyson's poem). There are no indications, however, that ancient Chinese arithmetic ever used its number systems for the purpose of elaborate computations like Babylonian mathematics. The mathematics of the Ten Classics is of a simple kind and does not seem to go beyond the limits of Egyptian mathematics. There is some trigonometry, notably in the "Hai-Tao" or "Sea Island" Classic, but since this is ascribed to the third century A.D., we may not exclude Western influence.

Chinese mathematics is in the exceptional position that its tradition has remained practically unbroken until recent years, so that we can study its position in the community somewhat better than in the case of Egyptian and Babylonian mathematics which belonged to vanished civilizations. We know, for instance, that candidates for examination had to display a precisely circumscribed knowledge of the Ten Classics and that this examination was based mainly on the ability to cite texts correctly from memory. The traditional lore was thus transmitted from generation to generation with painful conscientiousness. In such a stagnant cultural atmosphere new discoveries became extraordinary

exceptions, and this again guaranteed the invariability of the mathematical tradition. Such a tradition might be transmitted over millennia, only occasionally shaken by great historical catastrophes. In India a similar condition existed; here we even have examples of mathematical texts written in metric stanzas to facilitate memorization. There is no particular reason to believe that the ancient Egyptian and Babylonian practice may have been much different from the Indian and Chinese one. The emergence of an entirely new civilization was necessary to interrupt the complete ossification of mathematics. The different outlook on life characteristic of Greek civilization at last brought mathematics up to the standards of a real science.

Literature.

The Rhind Mathematical Papyrus, ed. T. E. Peet (London, 1923).

The Rhind Mathematical Papyrus, ed. by A. B. Chace, L. Bull, H. P. Manning and R. C. Archibald. (Oberlin, Ohio, 8 1927-29, 2 vols.).

This work includes an extensive bibliography of Egyptian and Babylonian mathematics. Another bibliography, mostly on ancient astronomy, in O. Neugebauer, l.c. p. 18.

Mathematischer Papyrus des staatlichen Museums der schönen Künste in Moskau, ed. by W. W. Struve and B. A. Turajeff (Berlin, 1930).

O. Neugebauer, *Vorlesungen über Geschichte der antiken mathematischen Wissenschaften, I. Vorgriechische Mathematik* (Berlin, 1934).

O. Neugebauer, *Mathematische Keilschrift-Texte* (Berlin, 1935-37, 3 vols).

O. Neugebauer—A. Sachs, *Mathematical cuneiform texts* (New Haven, 1945).

F. Thureau-Dangin, *Sketch of a history of the sexagesimal system,* Osiris 7 (1939), pp. 95–141.

F. Thureau-Dangin, *Textes mathématiques babyloniens* (Leiden, 1938).

There is some difference in the interpretation of Babylonian mathematics by the two preceding authors. An opinion is expressed by:

S. Gandz, *Conflicting Interpretations of Babylonian Mathematics,* Isis 31 (1940) pp. 405-425.

A good survey of pre-Greek mathematics in:

R. C. Archibald, *Mathematics Before the Greeks,* Science 71 (1930) pp. 109-121, 342, see also 72 (1930) p. 36.

D. E. Smith, *Algebra of 4000 Years Ago,* Scripta mathematica 4 (1936) pp. 111-125.

On Indian mathematics see the volumes of the Bulletin of the Calcutta Mathematical Society, and:

B. Datta–A. N. Simha, *History of Hindu Mathematics* (Lahore, 1935–1938, 2 vols.), (review by O. Neugebauer, Quellen u. Studien, vol.3B(1936), pp. 263–271).

G. R. Kaye, *Indian Mathematics*, Isis 2 (1919) pp. 326-356.

On Chinese-Japanese mathematics:

Y. Mikami, *The Development of Mathematics in China and Japan* (Leipzig, 1913).

D. E. Smith—Y. Mikami, *A History of Japanese Mathematics* (Chicago, 1914).

There is a translation of the *I-Ching, or Book of Changes*, by R. Wilhelm (New York, 1950).

On the nature of oriental society:

K. A. Wittfogel, *Die Theorie der orientalischen Gesellschaft*, Zeitschrift für Sozialforschung 7 (1938) pp. 90-122.

See also B. L. Van der Waerden, *l. c.* p. 82.

CHAPTER III

Greece

1. Enormous economic and political changes occurred in and around the Mediterranean basin during the last centuries of the second millennium. In a turbulent atmosphere of migrations and wars the Bronze Age was replaced by what has been called our Age, the Age of Iron. Few details are known about this period of revolutions, but we find that towards its end, perhaps circa 900 B.C., the Minoan and Hittite empires had disappeared, the power of Egypt and Babylonia had been greatly reduced, and new peoples had come into historical setting. The most outstanding of these new peoples were the Hebrews, the Assyrians, the Phoenicians, and the Greeks. The replacement of bronze by iron brought not only a change in warfare but by cheapening the tools of production increased the social surplus, stimulated trade, and allowed larger participation of the common people in matters of economy and public interest. This was reflected in two great innovations, the replacement of the clumsy script of the Ancient Orient by the easy-to-learn alphabet and the introduction of coined money, which helped to stimulate trade. The time had come when culture could no longer be the exclusive province of an Oriental officialdom.

The activities of the "sea-raiders", as some of the migrating peoples are styled in Egyptian texts, were originally accompanied by great cultural losses. The

Minoan civilization disappeared; Egyptian art declined; Babylonian and Egyptian science stagnated for centuries. No mathematical texts have come to us from this transition period. When stable relations were again established the Ancient Orient recovered mainly along traditional lines, but the stage was set for an entirely new type of civilization, the civilization of Greece.

The towns which arose along the coast of Asia Minor and on the Greek mainland were no longer administration centers of an irrigation society. They were trading towns in which the old-time feudal landlords had to fight a losing battle with an independent, politically conscious merchant class. During the Seventh and Sixth Centuries B.C. this merchant class won ascendancy and had to fight its own battles with the small traders and artisans, the demos. The result was the rise of the Greek *polis*, the self governing city state, a new social experiment entirely different from the early city states of Sumer and other Oriental countries. The most important of these city states developed in Ionia on the Anatolian coast. Their growing trade connected them with the shores of the whole Mediterranean, with Mesopotamia, Egypt, Scythia and even with countries beyond. Milete for a long time took a leading place. Cities on other shores also gained in wealth and importance: on the mainland of Greece first Corinth, later Athens; on the Italian coast Croton and Tarentum; in Sicily, Syracuse.

This new social order created a new type of man. The merchant trader had never enjoyed so much independence, but he knew that this independence was a result of a constant and bitter struggle. The static outlook of

the Orient could never be his. He lived in a period of geographical discoveries comparable only to those of Sixteenth Century Western Europe; he recognized no absolute monarch or power supposedly vested in a static Deity. Moreover, he could enjoy a certain amount of leisure, the result of wealth and of slave labor. He could philosophize about this world of his. The absence of any well established religion led many inhabitants of these coastal towns into mysticism, but also stimulated its opposite, the growth of rationalism and the scientific outlook.

2. Modern mathematics was born in this atmosphere of Ionian rationalism—the mathematics which not only asked the Oriental question "how?" but also the modern, the scientific question, "why?" The traditional father of Greek mathematics is the merchant Thales of Milete who visited Babylon and Egypt in the first half of the Sixth Century. And even if his whole figure is legendary, it stands for something eminently real. It symbolizes the circumstances under which the foundations not only of modern mathematics but also of modern science and philosophy were established.

The early Greek study of mathematics had one main goal, the understanding of man's place in the universe according to a rational scheme. Mathematics helped to find order in chaos, to arrange ideas in logical chains, to find fundamental principles. It was the most rational of all sciences, and though there is little doubt that the Greek merchants became acquainted with Oriental mathematics along their trade routes, they soon found out that the Orientals had left most of the rational-

ization undone. Why had the isosceles triangle two equal angles? Why was the area of a triangle equal to half that of a rectangle of equal base and altitude? These questions came naturally to men who asked similar questions concerning cosmology, biology, and physics.

It is unfortunate that there are no primary sources which can give us a picture of the early development of Greek mathematics. The existing codices are from Christian and Arabic times, and they are only sparingly supplemented by Egyptian papyrus notes of a somewhat earlier date. Classical scholarship, however, has enabled us to restore the remaining texts, which date back to the Fourth Century B.C. and later, and we possess in this way reliable editions of Euclid, Archimedes, Apollonios, and other great mathematicians of antiquity. But these texts represent an already fully developed mathematical science, in which historical development is hard to trace even with the aid of later commentaries. For the formative years of Greek mathematics we must rely entirely on small fragments transmitted by later authors and on scattered remarks by philosophers and other not strictly mathematical authors. Highly ingenious and patient text criticism has been able to elucidate many obscure points in this early history, and it is due to this work, carried on by investigators such as Paul Tannery, T. L. Heath, H. G. Zeuthen, E. Frank and others, that we are able to present something like a consistent, if largely hypothetical, picture of Greek mathematics in its formative years.

3. In the Sixth Century B.C. a new and vast Oriental

power arose on the ruins of the Assyrian Empire: the Persia of the Achaemenides. It conquered the Anatolian towns, but the social structure on the mainland of Greece was already too well established to suffer defeat. The Persian invasion was repelled in the historic battles of Marathon, Salamis, and Plataea. The main result of the Greek victory was the expansion and hegemony of Athens. Here, under Perikles in the second half of the Fifth Century, the democratic elements became increasingly influential. They were the driving force behind the economic and military expansion and made Athens by 430 not only the leader of a Greek Empire but also the center of a new and amazing civilization—the Golden Age of Greece.

Within the framework of the social and political struggles philosophers and teachers presented their theories and with them the new mathematics. For the first time in history a group of critical men, the "sophists," less hampered by tradition than any previous group of learned persons, approached problems of a mathematical nature in the spirit of understanding rather than of utility. As this mental attitude enabled the sophists to reach toward the foundations of exact thinking itself, it would be highly instructive to follow their discussions. Unfortunately, only one complete mathemathical fragment of this period is extant; it is written by the Ionian philosopher Hippokrates of Chios. This fragment represents a high degree of perfection in mathematical reasoning and deals, typically enough, with a curiously "impractical" but theoretically valuable subject, the so-called "lunulae"—the little moons or crescents bounded by two circular arcs.

The subject—to find certain areas bounded by two circular arcs which can be expressed rationally in terms of the diameters—has a direct bearing on the problem of the quadrature of the circle, a central problem in Greek mathematics. In the analysis of his problem[1] Hippokrates showed that the mathematicians of the Golden Age of Greece had an ordered system of plane geometry, in which the principle of logical deduction from one statement to another ("apagoge") had been fully accepted. A beginning of axiomatics had been made, as is indicated by the name of the book supposedly written by Hippokrates, the "Elements" ("stoicheia"), the title of all Greek axiomatic treatises including that of Euclid. Hippokrates investigated the areas of plane figures bounded by straight lines as well as circular arcs. The areas of similar circular segments, he teaches, are to each other as the squares of their chords. Pythagoras' theorem is known to him and so is the corresponding inequality for non-rectangular triangles. The whole treatise is already in what might be called the Euclidean tradition, but it is older than Euclid by more than a century.

The problem of the quadrature of the circle is one of the "three famous mathematical problems of antiquity," which in this period began to be a subject of study. These problems were:

(1) The trisection of the angle; that is, the problem of dividing a given angle into three equal parts.

(2) The duplication of the cube; that is, to find the side of a cube of which the volume is twice that of a given cube (the so-called Delic cube.)

(3) The quadrature of the circle; that is, to find the square of an area equal to that of a given square.

The importance of these problems lies in the fact that they cannot be geometrically solved by the construction of a finite number of straight lines and circles except by approximation, and therefore they served as a means of penetration into new fields of mathematics. They led to the discovery of the conic sections, and some cubic and quartic curves, and to one transcendental curve, the quadratrix. The anecdotic forms, in which the problems have occasionally been transmitted (Delphic oracles, etc.) should not prejudice us against their fundamental importance. It occurs not infrequently that a fundamental problem is presented in the form of an anecdote or a puzzle—Newton's apple, Cardan's broken promise, or Kepler's wine barrels. Mathematicians of different periods, including our own, have shown the connection between these Greek problems and the modern theory of equations, involving considerations concerning domains of rationality, algebraic numbers, and group theory.

4. Probably outside of the group of sophists, who were in some degree connected with the democratic movement, stood another group of mathematically inclined philosophers related to the aristocratic factions. They called themselves Pythagoreans after a rather mythical founder of the school, Pythagoras, supposedly a mystic, a scientist, and an aristocratic statesman.

[1]For a modern analysis see E. Landau, *Über quadrirbare Kreisbogenzweiecke*, Berichte Berliner Math. Ges. 2 (1903) pp. 1-6.

Where most sophists emphasized the reality of change, the Pythagoreans stressed the study of the unchangeable elements in nature and society. In their search for the eternal laws of the universe they studied geometry, arithmetic, astronomy, and music (the "quadrivium"). Their most outstanding leader was Archytas of Tarentum who lived about 400 and to whose school, if we follow the hypothesis of E. Frank, much of the "Pythagorean" brand of mathematics may be ascribed. Its arithmetic was a highly speculative science, which had little in common with the contemporary Babylonian computational technique. Numbers were divided into classes, odd, even, even-times-even, odd-times-odd, prime and composite, perfect, friendly, triangular, square, pentagonal, etc. Some of the most interesting results concern the "triangular numbers," which represent a link between geometry and arithmetic:

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                                  ·
                    ·            · ·
          ·        · ·          · · ·
 · 1,    · · 3,   · · · 6,     · · · · 10,   etc.
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Our name "square numbers" had its origin in Pythagorean speculations:

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                  · · ·
         · ·      · · ·
 · 1,    · · 4,   · · · 9,   etc.
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The figures themselves are much older, since some of them appear on neolithic pottery. The Pythagoreans investigated their properties, adding their brand of number mysticism and placing them in the center of a cosmic philosophy which tried to reduce all relations to number relations ("everything is number"). A point was "unity in position."

The Pythagoreans knew some properties of regular polygons and regular bodies. They showed how the plane can be filled by means of patterns of regular triangles, squares, or regular hexagons, and space by cubes, to which Aristotle later tried to add the wrong notion that space can be filled by regular tetrahedra. The Pythagoreans may have also known the regular oktahedron and dodekahedron——the latter figure because pyrite, found in Italy, crystallizes in dodekahedra, and models of such figures as ornaments or magical symbols date back to Etruscan times.

As to Pythagoras' theorem, the Pythagoreans ascribed its discovery to their master who was supposed to have sacrificed a hundred oxen to the gods as a token of gratitude. We have seen that the theorem was already known in Hammurabi's Babylon, but the first general proof may very well have been obtained in the Pythagorean school.

The most important discovery ascribed to the Pythagoreans was the discovery of the irrational by means of incommensurable line segments. This discovery may have been the result of their interest in the geometric mean $a : b = b : c$, which served as a symbol of aristocracy. What was the geometric mean of 1 and 2, two sacred symbols? This led to the study of the ratio of the side and the diagonal of a square, and it was found that this ratio could not be expressed by "numbers"—that is, by what we now call rational numbers (integers or fractions), the only numbers recognized as such.

Suppose that this ratio be $p : q$, in which we can always take numbers p and q relative prime. Then $p^2 = 2q^2$, hence p^2, and therefore p, is even, say $p = 2r$. Then q must be odd, but since $q^2 = 2r^2$, it must also be even. This contradiction was not solved, as in the Orient or in Renaissance Europe, by an extension of the conception of number, but by rejecting the theory of numbers for such cases and looking for a synthesis in geometry.

This discovery, which upset the easy harmony between arithmetics and geometry, was probably made in the last decades of the Fifth Century B.C. It came on top of another difficulty, which had emerged from the arguments concerning the reality of change, arguments which have kept philosophers busy from then until the present. This difficulty has been ascribed to Zeno of Elea (c. 450 B.C.), a pupil of Parmenides, a conservative philosopher who taught that reason only recognizes absolute being, and that change is only apparent. It received mathematical significance when infinite processes had to be studied in such questions as the determination of the volume of a pyramid. Here Zeno's paradoxes came in conflict with some ancient and intuitive conceptions concerning the infinitely small and the infinitely large. It was always believed that the sum of an infinite number of quantities can be made as large as we like, even if each quantity is extremely small ($\infty \times \epsilon = \infty$), and also that the sum of a finite or infinite number of quantities of dimension zero is zero ($n \times 0 = 0$, $\infty \times 0 = 0$). Zeno's criticism challenged these conceptions and his four paradoxes created a stir of which the ripples can be observed to-day. They have been preserved by Aristotle and are known as the

Achilles, the Arrow, the Dichotomy, and the Stadium. They are phrased so as to stress contradictions in the conception of motion and of time; no attempt is made to solve the contradictions.

The gist of the reasoning will be clear from the Achilles and the Dichotomy, which we explain in our own words as follows:

Achilles. Achilles and a tortoise move in the same direction on a straight line. Achilles is much faster than the tortoise, but in order to reach the tortoise he must first pass the point P from which the tortoise started. If he comes to P, the tortoise has advanced to point P_1. Achilles cannot reach the tortoise until he passed P_1, but the tortoise has advanced to a new point P_2. If Achilles is at P_2, the tortoise has reached a new point P_3, etc. Hence Achilles can never reach the tortoise.

Dichotomy. Suppose I like to go from A to B along a line. In order to reach B, I must first traverse half the distance AB_1 of AB, and in order to reach B_1 I must first reach B_2 half way between A and B_1. This goes on indefinitely, so that the motion can never even *begin*.

Zeno's arguments showed that a finite segment can be broken up into an infinite number of small segments each of finite length. They also showed that there is a difficulty in explaining what we mean by saying that a line is "composed of" points. It is very likely that Zeno himself had no idea of the mathematical implications of his arguments. Problems leading to his paradoxes have regularly appeared in the course of philosophical and theological discussions; we recognize them as problems concerning the relation of the potentially and the actually infinite. Paul Tannery, however, believed that Zeno's arguments were particularly directed

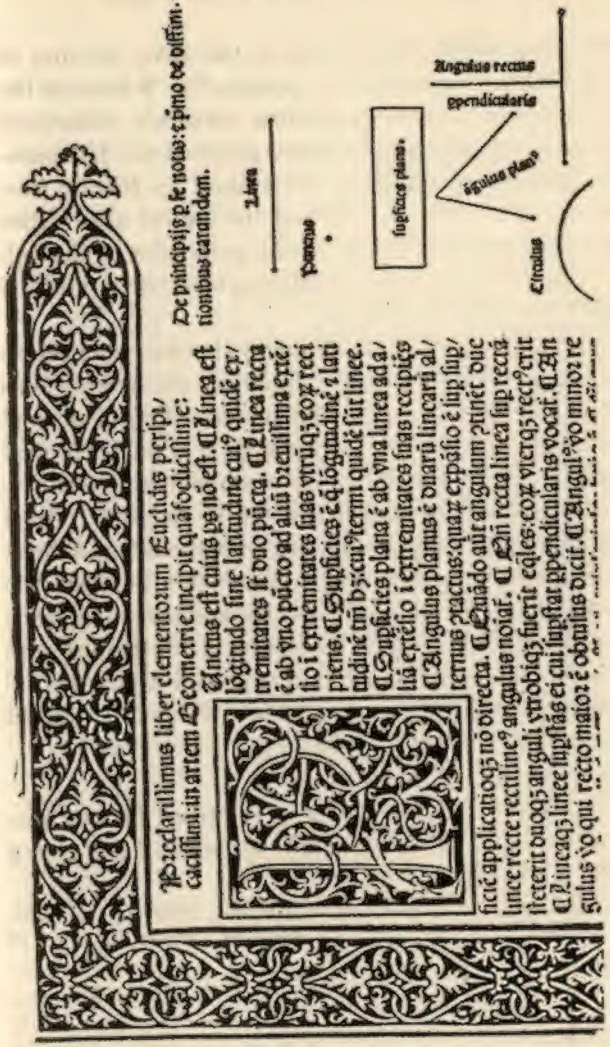
against the Pythagorean idea of the space as sum of points ("the point is unity in position")[1]. Whatever the truth may be, Zeno's reasoning certainly influenced mathematical thought for many generations. His paradoxes may be compared to those used by Bishop Berkeley in 1734, when he showed the logical absurdities to which poor formulation of the principles of the calculus may lead, but without offering a better foundation himself.

Zeno's arguments began to worry the mathematicians even more after the irrational had been discovered. Was mathematics as an exact science possible? Tannery[2] has suggested that we may speak of "a veritable logical scandal"—of a *crisis* in Greek mathematics. If this is the case, then this crisis originated in the later period of the Peloponnesian war, ending with the fall of Athens (404). We may then detect a connection between the crisis in mathematics and that of the social system, since the fall of Athens spelled the doom of the empire of a slave-owning democracy and introduced a new period of aristocratic supremacy—a crisis which was solved in the spirit of the new period.

5. Typical of this new period in Greek history was the increasing wealth of certain sections of the ruling

[1] P. Tannery, *La géométrie grecque* (Paris, 1887) pp. 217-261. Another opinion in B. L. Van der Waerden, Math Annalen 117 (1940) pp. 141-161.

[2] P. Tannery, *La géométrie grecque* (Paris, 1887) p. 98. Tannery, at this place, deals only with the breakdown of the ancient theory of proportions as a result of the discovery of incommensurable line segments.



PORTION OF A PAGE OF THE FIRST EDITION OF EUCLID'S *Elements*, 1482

classes combined with equally increased misery and insecurity of the poor. The ruling classes based their material existence more and more upon slavery, which allowed them leisure to cultivate arts and sciences, but made them also more and more averse to all manual work. A gentleman of leisure looked down upon work which slaves and craftsmen did and sought relief from worry in the study of philosophy and of personal ethics. Plato and Aristotle expressed this attitude; and it is in Plato's "Republic" (written perhaps c. 360) that we find the clearest expression of the ideals of the slave-owning aristocracy. The "guards" of Plato's republic must study the "quadrivium," consisting of arithmetic, geometry, astronomy, and music, in order to understand the laws of the universe. Such an intellectual atmosphere was conducive (at any rate in its earlier period) to a discussion of the foundations of mathematics and to speculative cosmogony. At least three great mathematicians of this period were connected with Plato's Academy, namely, Archytas, Theaetetos (d. 369), and Eudoxos (c. 408-355). Theaetetos has been credited with the theory of irrationals as it appears in the tenth book of Euclid's "Elements." Eudoxos' name is connected with the theory of proportions which Euclid gave in his fifth book, and also with the so-called "exhaustion" method, which allowed a rigorous treatment of area and volume computations. This means that it was Eudoxos who solved the "crisis" in Greek mathematics, and whose rigorous formulations helped to decide the course of Greek axiomatics and, to a considerable extent, of Greek mathematics as a whole.

Eudoxos' theory of proportions did away with the arithmetical theory of the Pythagoreans, which applied to commensurable quantities only. It was a purely geometrical theory, which, in strictly axiomatic form, made all reference to incommensurable or commensurable magnitudes superfluous.

Typical is Def. 5, Book V of Euclid's "Elements":

"Magnitudes are said to be in the same ratio, the first to the second and the third to the fourth, when, if any equimultiples whatever be taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or are alike less than, the latter equimultiples taken in corresponding order".

The present theory of irrational numbers, developed by Dedekind and Weierstrass, follows Eudoxos' mode of thought almost literally but by using modern arithmetical methods has opened far wider perspectives.

The "exhaustion method" (the term "exhaust" appears first in Grégoire de Saint Vincent, 1647) was the Platonic school's answer to Zeno. It avoided the pitfalls of the infinitesimal by simply discarding them, by reducing problems which might lead to infinitesimals to problems involving formal logic only. When, for instance, it was required to prove that the volume V of a tetrahedron is equal to one third the volume P of a prism of equal base and altitude, the proof consisted in showing that both the assumptions $V > 1/3\ P$ and $V < 1/3\ P$ lead to absurdities. For this purpose an axiom was introduced, now known as the axiom of Archimedes, and which also underlies Eudoxos' theory of

proportions, namely, that "magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another" (Euclid V, Def. 4)[1]. This method, which became the standard Greek and Renaissance mode of strict proof in area and volume computation, was quite rigorous, and can easily be translated into a proof satisfying the requirements of modern analysis. It had the great disadvantage that the result, in order to be proved, must be known in advance, so that the mathematician finds it first by some other, less rigorous and more tentative method.

There are clear indications that such an other method was actually used. We possess a letter from Archimedes to Erathostenes (c. 250 B.C.), which was not discovered until 1906, in which Archimedes described a non-rigorous but fertile way of finding results. This letter is known as the "Method". It has been suggested, notably by S. Luria, that it represented a school of mathematical reasoning competing with the school of Eudoxos, also dating back to the period of the "crisis" and associated with the name of Democritus, the founder of the atom theory. In Democritus' school, according to the theory of Luria, the notion of the "geometrical atom" was introduced. A line segment, an area, or a volume was supposed to be built up of a large, but finite, number of indivisible "atoms." The computation of a volume was the summation of the

[1]Archimedes' version (which he explicitly attributes to Eudoxos) is: "When two spaces are unequal, then it is possible to add to itself the difference by which the lesser is surpassed by the greater, so often that every finite space will be exceeded" (in *On the Sphere and the Cylinder*).

volumes of all the "atoms" of which this body consists. This theory sounds perhaps absurd, until we realise that several mathematicians of the period before Newton, notably Viète and Kepler, used essentially the same conceptions, taking the circumference of a circle as composed of a very large number of tiny line segments. There is no evidence that antiquity ever developed a rigorous method on this foundation, but our modern limit conceptions have made it possible to build this "atom" theory into a theory as rigorous as the exhaustion method. Even today we use this conception of the "atoms" quite regularly by setting up a mathematical problem in the theory of elasticity, in physics or in chemistry, reserving the rigorous "limit" theory to the professional mathematician[1].

The advantage of the "atom" method over the "exhaustion" method was that it facilitated the finding of new results. Antiquity had thus the choice between a rigorous but relatively sterile, and a loosely founded but far more fertile, method. It is instructive that in practically all the classical texts the first method was used. This again may have a connection with the fact that mathematics had become a hobby of a leisure class, basing itself on slavery, indifferent to invention, and interested in contemplation. It may also be a reflection of the victory of Platonic idealism over Demokritian

[1]"Thus, so far as first differentials are concerned, a small part of a curve near a point may be considered straight and a part of a surface plane; during a short time a particle may be considered as moving with constant velocity and any physical process as occurring at a constant rate". (H. B. Phillips, *Differential Equations*, London, 1922, p. 7.)

materialism in the realm of mathematical philosophy.

6. In 334 Alexander the Great began his conquest of Persia. When he died at Babylon in 323 the whole Near East had fallen to the Greeks. Alexander's conquests were divided among his generals and eventually three empires emerged: Egypt under the Ptolemies; Mesopotamia and Syria under the Seleucids; and Macedonia under Antigonos and his successors. Even the Indus valley had its Greek princes. The period of Hellenism had begun.

The immediate consequence of Alexander's campaign was that the advance of Greek civilization over large sections of the Oriental world was accelerated. Egypt and Mesopotamia and a part of India were Hellenized. The Greeks flooded the Near East as traders, merchants, physicians, adventurers, travellers, and mercenaries. The cities—many of them newly founded and recognizable by their Hellenistic names—were under Greek military and administrative control, and had a mixed population of Greeks and Orientals. But Hellenism was essentially an urban civilization. The country-side remained native and continued its existence in the traditional way. In the cities the ancient Oriental culture met with the imported civilization of Greece and partly mixed with it, though there always remained a deep separation between the two worlds. The Hellenistic monarchs adopted Oriental manners, had to deal with Oriental problems of administration, but stimulated Greek arts, letters, and sciences.

Greek mathematics, thus transplanted to new surroundings, kept many of its traditional aspects, but experienced also the influence of the problems in administration and astronomy which the Orient had to solve. This close contact of Greek science with the Orient was extremely fertile, especially during the first centuries. Practically all the really productive work which we call "Greek mathematics" was produced in the relatively short interval from 350 to 200 B.C., from Eudoxos to Apollonius, and even Eudoxos' achievements are only known to us through their interpretation by Euclid and Archimedes. And it is also remarkable that the greatest flowering of this Hellenistic mathematics occurred in Egypt under the Ptolemies and not in Mesopotamia, despite the more advanced status of native mathematics in Babylonia.

The reason for this development may be found in the fact that Egypt was now in a central position in the Mediterranean world. Alexandria, the new capital, was built on the sea coast, and became the intellectual and economic center of the Hellenistic world. But Babylon lingered on only as a remote center of caravan roads, and eventually disappeared to be replaced by Ktesiphon-Seleucia, the new capital of the Seleucids. No great Greek mathematicians, as far as we know, were ever connected with Babylon. Antioch and Pergamum, also cities of the Seleucid Empire but closer to the Mediterranean, had important Greek schools. The development of native Babylonian astronomy and mathematics even reached its height under the Seleucids, and Greek astronomy received an impetus, the importance of which is only now beginning to be better understood. Beside Alexandria there were some other centers of mathematical learning, especially Athens and Syracuse.

Athens became an educational center, while Syracuse produced Archimedes, the greatest of Greek mathematicians.

7. In this period the professional scientist appeared, a man who devoted his life to the pursuit of knowledge and received a salary for doing it. Some of the most outstanding representatives of this group lived in Alexandria, where the Ptolemies built a great center of learning in the so-called Museum with its famous Library. Here the Greek heritage in science and literature was preserved and developed. The success of this enterprise was considerable. Among the first scholars associated with Alexandria was Euclid, one of the most influential mathematicians of all times.

Euclid, about whose life nothing is known with any certainty, lived probably during the time of the first Ptolemy (306-283), to whom he is supposed to have remarked that there is no royal road to geometry. His most famous and most advanced texts are the thirteen books of his "Elements" ("stoicheia"), though he is also credited with several other minor texts. Among these other texts are the "Data," containing what we would call applications of algebra to geometry but presented in strictly geometrical language. We do not know how many of these texts are Euclid's own and how many are compilations, but they show at many places an astonishing penetration. They are the first full mathematical texts that have been preserved from Greek antiquity.

The "Elements" form, next to the Bible, probably the book most reproduced and studied in the history of

the Western world. More than a thousand editions have appeared since the invention of printing and before that time manuscript copies dominated much of the teaching of geometry. Most of our school geometry is taken, often literally, from six of the thirteen books; and the Euclidean tradition still weighs heavily on our elementary instruction. For the professional mathematician these books have always had an inescapable fascination, and their logical structure has influenced scientific thinking perhaps more than any other text in the world.

Euclid's treatment is based on a strictly logical deduction of theorems from a set of definitions, postulates, and axioms. The first four books deal with plane geometry and lead from the most elementary properties of lines and angles to the congruence of triangles, the equality of areas, the theorem of Pythagoras (I, 47), the construction of a square equal to a given rectangle, the golden section, the circle, and the regular polygons. The fifth book presents Eudoxos' theory of incommensurables in its purely geometrical form, and in the sixth book this is applied to the similarity of triangles. This introduction of similarity at such a late stage is one of the most important differences between Euclid's presentation of plane geometry and the present one, and must be ascribed to the emphasis laid by Euclid on Eudoxos' novel theory of incommensurables. The geometrical discussion is resumed in the tenth book, often considered Euclid's most difficult one, which contains a geometrical classification of quadratic irrationals and of their quadratic roots, hence of what we call numbers of a form $\sqrt{a + \sqrt{b}}$. The last three books deal with

solid geometry and lead via solid angles, the volumes of parallelepipeds, prisms, and pyramids to the sphere and to what seems to have been intended as the climax: the discussion of the five regular ("Platonic") bodies and the proof that only five such bodies exist.

Books VII-IX are devoted to number theory—not to computational technique but to such Pythagorean subjects as the divisibility of integers, the summation of the geometrical series, and some properties of prime numbers. There we find both "Euclid's algorithm" to find the greatest common divisor of a given set of numbers, and "Euclid's theorem" that there are an infinite number of primes (IX, 20). Of particular interest is theorem VI, 27, which contains the first maximum problem that has reached us, with the proof that the square, of all rectangles of given perimeter, has maximum area. The fifth postulate of Book I (the relation between "axioms" and "postulates" in Euclid is not clear) is equivalent to the so-called "parallel axiom", according to which one and only one line can be drawn through a point parallel to a given line. Attempts to reduce this axiom to a theorem led in the Nineteenth Century to a full appreciation of Euclid's wisdom in adopting it as an axiom and to the discovery of other, so-called non-euclidean geometries.

Algebraic reasoning in Euclid is cast entirely into geometrical form. An expression $\sqrt{A}$ is introduced as the side of a square of area A, a product ab as the area of a rectangle with sides a and b. This mode of expression was primarily due to Eudoxos' theory of proportions, which consciously rejected numerical expressions for line segments and in this way dealt with incommensurables in a purely geometrical way—"numbers" being conceived only as integers or rational fractions.

What was Euclid's purpose in writing his "Elements"? We may assume with some confidence that he wanted to bring together into one text three great discoveries of the recent past: Eudoxos' theory of proportions, Theaetetos' theory of irrationals, and the theory of the five regular bodies which occupied an outstanding place in Plato's cosmology. These were all three typically "Greek" achievements.

8. The greatest mathematician of the Hellenistic period—and of antiquity as a whole—was Archimedes (287-212) who lived in Syracuse as adviser to King Hiero. He is one of the few scientific figures of antiquity who is more than a name; several data about his life and person have been preserved. We know that he was killed when the Romans took Syracuse, after he placed his technical skill at the disposal of the defenders of the city. This interest in practical applications strikes us as odd if we compare it to the contempt in which such interest was held in the Platonic school of his contemporaries, but an explanation is found in the much quoted statement in Plutarch's "Marcellus", that

> "although these inventions had obtained for him the reputation of more than human sagacity, he did not deign to leave behind any written work on such subjects, but, regarding as ignoble and sordid the business of mechanics and every sort of art which is directed to use and profit, he placed his whole ambition in those speculations the beauty and subtlety of which are untainted by any admixture of the common needs of life".

σονα ἢ ὃν ,γιγ ∟′ δ′ πρὸς ψπ. δίχα ἡ ὑπὸ *ΓΑΗ* τῇ *ΑΘ*· ἡ *ΑΘ* ἄρα διὰ τὰ αὐτὰ πρὸς τὴν *ΘΓ* ἐλάσσονα λόγον ἔχει ἢ ὃν ,ε϶κδ ∟′ δ′ πρὸς ψπ ἢ ὃν ,αωκγ πρὸς σμ· ἑκατέρα γὰρ ἑκατέρας δ ιγ′· ὥστε ἡ *ΑΓ* πρὸς τὴν *ΓΘ* ἢ ὃν ,αωλη θ ια′ πρὸς σμ. ἔτι δίχα ἡ ὑπὸ *ΘΑΓ* τῇ *ΚΑ*· καὶ ἡ *ΑΚ* πρὸς τὴν *ΚΓ* ἐλάσσονα [ἄρα] λόγον ἔχει ἢ ὃν ,αξ πρὸς ξϛ· ἑκατέρα γὰρ ἑκατέρας ια μ′· ἡ *ΑΓ* ἄρα πρὸς [τὴν] *ΚΓ* ἢ ὃν ,αθ ϛ′ πρὸς ξϛ. ἔτι δίχα ἡ ὑπὸ *ΚΑΓ* τῇ *ΛΑ*· ἡ *ΑΛ* ἄρα πρὸς [τὴν] *ΛΓ* ἐλάσσονα λόγον ἔχει ἢ ὃν τὰ ,βιϛ ϛ′ πρὸς ξϛ, ἡ δὲ *ΑΓ* πρὸς *ΓΛ* ἐλάσσονα ἢ τὰ ,βιζ δ′ πρὸς ξϛ. ἀνάπαλιν ἄρα ἡ περίμετρος τοῦ πολυγώνου πρὸς τὴν διάμετρον μείζονα λόγον ἔχει ἤπερ ,ϛτλϛ πρὸς ,βιζ δ′, ἅπερ τῶν ,βιζ δ′ μείζονά ἐστιν ἢ τριπλασίονα καὶ δέκα οα′· καὶ ἡ περίμετρος ἄρα τοῦ ϟϛγώνου τοῦ ἐν τῷ κύκλῳ τῆς διαμέτρου τριπλασίων ἐστὶ καὶ μείζων ἢ ι οα′· ὥστε καὶ ὁ κύκλος ἔτι μᾶλλον τριπλασίων ἐστὶ καὶ μείζων ἢ ι οα′.

ἡ ἄρα τοῦ κύκλου περίμετρος τῆς διαμέτρου τριπλασίων ἐστὶ καὶ ἐλάσσονι μὲν ἢ ἑβδόμῳ μέρει, μείζονι δὲ ἢ ι οα′ μείζων.

1 ∟′] Eutocius, γ′ AB(C). 3 ,ε϶κδ] Eutocius, e corr. B, ,ετκδ ABC. ∟′] Eutocius, e corr. B, ε′ A, 3 B. 4 σμ] B²C, σν AB. ιγ′] B², ιγ′ α′ A(C); δ ιγ′ om. B. 5 ια′] B², om. AB(C). 7 ξϛ] C, e corr. B, σξϛ AB. 8 ἑκατέρας] B², εκατερα ABC. ια μ′· ἡ *ΑΓ*] B², *Wallis*, οιμαι AB, οιμ(·) C. πρὸς ΓΚ Eutocius. *ΚΓ* ἢ ὃν] B², *Wurmius*; (ΓΚ) . . (χε)ν C, καταγον A. ,αθ ϛ′] B²C, αοϛ A. 10 *ΛΓ*] *Wallis*, *ΑΓ* ABC; πρὸς *ΛΓ* Eutocius. 13 ,ϛτλϛ] Eutocius, B², *Wallis*, ,ϛτα ϛ′ ABC. 14 ,βιζ (pr.)] e corr. B, ,ζιζ AC. 15 οα′] B, corr. ex ο′α′ C, ο′α′ A. 16 ϟϛγώνου] C, ϟϛ πολυγωνου AB. 17 ι οα′] e corr. B, ὃν ο′ ια′ AB(C). 18 ι οα′] e corr. B, θ′ ια′ AC. 20 ἐλάσσονι] scripsi, ελασσων ABC. μείζονι—21 μείζων] scripsi,

REPRODUCTION OF TWO PAGES OF *Archimedes Opera Omnia* (computation of π) (Edited by J. Heiberg, Vol. I, 1910)

$$A\Gamma : \Gamma H < 3013\tfrac{1}{2}\,\tfrac{1}{4} : 780 \text{ [u. Eutocius].}$$

secetur $\angle\, \Gamma A H$ in duas partes aequales recta $A\Theta$; propter eadem igitur erit $A\Theta : \Theta\Gamma < 5924\frac{1}{2}\,\frac{1}{4} : 780$ [u. Eutocius] siue $< 1823 : 240$; altera[1]) enim alterius $\frac{4}{13}$ est [u. Eutocius]; quare $A\Gamma : \Gamma\Theta < 1838\frac{9}{11} : 240$ [u. Eutocius]. porro secetur $\angle\, \Theta A \Gamma$ in duas partes aequales recta KA; est igitur $AK : K\Gamma < 1007 : 66$ [u. Eutocius]; altera[1]) enim alterius est $\frac{11}{40}$; itaque

$$A\Gamma : \Gamma K < 1009\tfrac{1}{6} : 66 \text{ [u. Eutocius].}$$

porro secetur $\angle\, K A \Gamma$ in duas partes aequales recta ΛA; erit igitur

$$A\Lambda : \Lambda\Gamma < 2016\tfrac{1}{6} : 66 \text{ [u. Eutocius],}$$

et $A\Gamma : \Gamma\Lambda < 2017\frac{1}{4} : 66$ [u. Eutocius]. et e contrario ⟨$\Gamma\Lambda : A\Gamma > 66 : 2017\frac{1}{4}$ [Pappus VII, 49 p. 688]. sed $\Gamma\Lambda$ latus est polygoni 96 latera habentis; quare⟩[2]) perimetrus polygoni ad diametrum maiorem rationem habet quam $6336 : 2017\frac{1}{4}$, quae maiora sunt quam triplo et $\frac{10}{71}$ maiora quam $2017\frac{1}{4}$; itaque etiam perimetrus polygoni inscripti 96 latera habentis maior est quam triplo et $\frac{10}{71}$ maior diametro; quare etiam multo magis[3]) circulus maior est quam triplo et $\frac{10}{71}$ maior diametro.

ergo ambitus circuli triplo maior est diametro et excedit spatio minore quam $\frac{1}{7}$, maiore autem quam $\frac{10}{71}$.

1) Exspectaueris ἑκάτερος (sc. ὅρος) γὰρ ἑκατέρου (ἑκάτερα γὰρ ἑκατέρων Wallis), sed genus femininum minus adcurate refertur ad auditum uerbum εὐθεῖα, quasi sit $A\Theta = 5924\frac{1}{2}\,\frac{1}{4}$, $\Theta\Gamma = 780$.

2) Ueri simile est, Archimedem ipsum haec addidisse. similes omissiones durae inueniuntur p. 240, 4, 6; 242, 5, 8, nec dubito eas transscriptori tribuere, sicut etiam p. 240, 8 τὸ πολύγωνον pro ἡ περίμετρος τοῦ πολυγώνου, p. 242, 17 ὁ κύκλος pro ἡ περίμετρος τοῦ κύκλου.

3) Quippe quae maior sit perimetro polygoni (De sph. et cyl. I p. 10, 1).

μειζων δε AC, maior B, autem quam decem septuagesimunis add. B². In fine: Ἀρχιμηδους κυκλου μετρησις A.

The most important contributions which Archimedes made to mathematics were in the domain of what we now call the integral calculus,—theorems on areas of plane figures and on volumes of solid bodies. In his "Measurement of the Circle" he found an approximation of the circumference of the circle by the use of inscribed and circumscribed regular polygons. Extending his approximation to polygons of 96 sides he found (in our notation):

$$3\frac{10}{71} < 3\,\frac{284\frac{1}{4}}{2018\frac{7}{40}} < 3\,\frac{284\frac{1}{4}}{2017\frac{1}{4}} < \pi < 3\,\frac{667\frac{1}{2}}{4673\frac{1}{2}}$$

$$< 3\,\frac{667\frac{1}{2}}{4672\frac{1}{2}} = 3\frac{1}{7}\ [1]$$

which is usually expressed by saying that π is about equal to $3\frac{1}{7}$. In Archimedes' book "On the Sphere and Cylinder" we find the expression for the area of the sphere (in the form that the area of a sphere is four times that of a great circle) and for the volume of a sphere (in the form that this volume is equal to 2/3 of the volume of the circumscribed cylinder). Archimedes' expression for the area of a parabolic segment (4/3 the area of the inscribed triangle with the same base as the segment and its vertex at the point where the tangent is parallel to the base) is found in his book on "The Quadrature of the Parabola." In the book on "Spirals" we find the "Spiral of Archimedes", with area computations; in "On Conoids and Spheroids" we find the volumes of certain quadratic surfaces of revolution. Archimedes' name is also connected with his theorem on the loss of weight of bodies submerged in a liquid, which can be found in his book "On Floating Bodies," a treatise on hydrostatics.

[1] $3.1409 < \pi < 3.1429$. The arithmetic mean of upper and lower limit gives $\pi = 3.1419$. The correct value is 3.14159.....

In all these works Archimedes combined a surprising originality of thought with a mastery of computational technique and rigor of demonstration. Typical of this rigor is the "axiom of Archimedes" already quoted and his consistent use of the exhaustion method to prove the results of his integration. We have seen how he actually found these results in a more heuristic way (by "weighing" infinitesimals); but he subsequently published them in accordance with the strictest requirements of rigor. In his computational proficiency Archimedes differed from most of the productive Greek mathematicians. This gave his work, with all its typically Greek characteristics, a touch of the Oriental. This touch is revealed in his "Cattle Problem," a very complicated problem in indeterminate analysis which may be interpreted as a problem leading to an equation of the "Pell" type:

$$t^2 - 4729494\,u^2 = 1,$$

which is solved by very large numbers.

This is only one of many indications that the Platonic tradition never entirely dominated Hellenistic mathematics; Hellenistic astronomy points in the same direction.

9. With the third great Hellenistic mathematician, Apollonios of Perga (c. 260-c. 170) we are again entirely within the Greek geometrical tradition. Apollonios, who seems to have taught at Alexandria and at Pergamum, wrote a treatise of eight books on "Conics," of which seven have survived, three only in an Arabic translation. It is a treatise on the ellipse, parabola, and hyperbola, introduced as sections of a circular cone, and penetrates as far as the discussion of the evolutes of a conic. We know these conics by the names found in Apollonios; they refer to certain area properties of these curves, which are expressed in our notation by the equations (homogeneous notation, p, d are lines in Apollonios) $y^2 = px$; $y^2 = px \pm \frac{p}{d}x^2$ (the plus gives the hyperbola, the minus the ellipse). Parabola here means "application," ellipse "application with deficiency," hyperbola "application with excess." Apollonios did not have our coordinate method because he had no algebraic notation (probably rejecting it consciously under influence of the school of Eudoxos). Many of his results, however, can be transcribed immediately into coordinate language—including his property of the evolutes, which is identical with the Cartesian equation.[1] This can also be said for other books by Apollonios, of which parts have been preserved and which contain "algebraic" geometry in geometrical and therefore homogeneous language. Here we find Apollonios' tangency problem which requires the construction of the circles tangent to three given circles; the circles may be replaced by straight lines or points. In Apollonios we meet for the first time in explicit form the requirement that geometrical constructions be confined to compass and ruler only, which therefore was not such a general "Greek" requirement as is sometimes believed.

[1] "My thesis, then, is that the essence of analytic geometry is the study of loci by means of their equations, and that this was known to the Greeks and was the basis of their study in conic sections." J. L. Coolidge, *A History of Geometrical Methods* (Oxford, 1940) p. 119. See, however, our remarks on Descartes.

10. Mathematics, throughout its history and until modern times, cannot be separated from astronomy. The needs of irrigation and of agriculture in general—and to a certain extent also of navigation—accorded to astronomy the first place in Oriental and in Hellenistic science, and its course determined to no small extent that of mathematics. The computational and often also the conceptual content of mathematics was largely conditioned by astronomy, and the progress of astronomy depended equally on the power of the mathematical books available. The structure of the planetary system is such that relatively simple mathematical methods allow far reaching results, but are at the same time complicated enough to stimulate improvement of these methods and of the astronomical theories themselves. The Orient itself had made considerable advances in computational astronomy during the period just preceeding the Hellenistic era, especially in Mesopotamia during the late Assyrian and Persian periods. Here observations consistently carried on over a long period had allowed a remarkable understanding of many ephemerides. The motion of the moon was one of the most challenging of all astronomical problems to the

mathematician, in antiquity as well as in the Eighteenth Century, and Babylonian ("Chaldean") astronomers devoted much effort to its study. The meeting of Greek and Babylonian science during the Seleucid period brought great computational and theoretical advancement, and where Babylonian science continued in the ancient calendaric tradition, Greek science produced some of its most significant theoretical triumphs.

The oldest known Greek contribution to theoretical astronomy was the planetary theory of the same Eudoxos who inspired Euclid. It was an attempt to explain the motion of the planets (around the earth) by assuming the super-position of four rotating concentric spheres, each with its own axis of rotation with the ends fixed on the enclosing sphere. This was something new, and typically Greek, an explanation rather than a chronicle of celestial phenomena. Despite its crude form Eudoxos' theory contained the central idea of all planetary theories until the Seventeenth Century, which consisted in the explanation of irregularities in the apparent orbits of moon and planets by the superposition of circular movements. It still underlies the computational side of our modern dynamic theories as soon as we introduce Fourier series.

Eudoxos was followed by Aristarchos of Samos (c. 280 B.C.), the "Copernicus of antiquity," credited by Archimedes with the hypothesis that the sun, and not the earth, is the center of the planetary motion. This hypothesis found few adherents in antiquity, though the belief that the earth rotates about its axis had a wide acceptance. The small success of the heliocentric hypothesis was mainly due to the authority of Hipparchos, often considered the greatest astronomer of antiquity.

Hipparchos of Nicaea made observations between 161-126 B.C. Little of his work has come to us directly, the main source of our knowledge of his achievements coming through Ptolemy who lived three centuries later. Much of the contents of Ptolemy's great work, the "Almagest," may be ascribed to Hipparchos, especially the use of eccentric circles and epicycles to explain the motion of sun, moon, and planets, as well as the discovery of the precession of the equinoxes. Hipparchos is also credited with a method to determine latitude and longitude by astronomical means, but antiquity never was able to muster a scientific organization sufficient to do any large scale mapping. (Scientists in antiquity were very thinly scattered, both in locality and in time.) Hipparchos' work was closely connected with the achievements of Babylonian astronomy, which reached great heights in this period; and we may see in this work the most important scientific fruit of the Greek—Oriental contact of the Hellenistic period.[1]

11. The third and last period of antique society is that of the Roman domination. Syracuse fell to Rome in 212, Carthage in 146, Greece in 146, Mesopotamia in 64, Egypt in 30 B.C. The whole Roman-dominated Orient, including Greece, was reduced to the status of a colony ruled by Roman administrators. This control did not affect the economic structure of the Oriental countries as long as the heavy taxes and other levies were duly delivered. The Roman Empire naturally

[1] O. Neugebauer, *Exact Science in Antiquity*, Studies in Civilization, Un. of Pennsylvania Bicentennial Conf. (Phila., 1942) pp. 22-31.

split into a Western part with extensive agriculture fitted for wholesale slavery, and an Eastern part with intensive agriculture which never used slaves except for domestic duties and for public works. Despite the growth of some cities and a commerce embracing the whole of the known Western world, the entire economic structure of the Roman Empire remained based on agriculture. The spread of a slave economy in such a society was fatal to all original scientific work. Slave owners as a class are seldom interested in technical discoveries, partly because slaves can do all the work cheaply, and partly because they fear to give any tool into the hands of slaves which may sharpen their intelligence. Many members of the ruling class dabbled in the arts and sciences, but this very dabbling promoted mediocrity rather than productive thinking. When with the decline of the slave market Roman economy declined, there were few men to cultivate even the mediocre science of the past centuries.

As long as the Roman Empire showed some stability, Eastern science continued to flourish as a curious blend of Hellenistic and Oriental elements. Though originality and stimulation gradually disappeared, the *pax Romana* lasting for many centuries allowed undisturbed speculation along traditional lines. Coexistent with the *pax Romana* was for some centuries the *pax Sinensis;* the Eurasian continent in all its history never knew such a period of uninterrupted peace as under the Antonins in Rome and the Han in China. This made the diffusion of knowledge over the continent from Rome and Athens to Mesopotamia, China, and India easier than ever before. Hellenistic science continued to flow into China

and India and was influenced in its turn by the science of those countries. Glimpses of Babylonian astronomy and Greek mathematics came to Italy, Spain, and Gaul, —an example is the spread of the sexagesimal division of angle and hour over the Roman Empire. There exists a theory of E. Woepcke which traces the spread of the so-called Hindu-Arabic numerals over Europe to Neo-Pythagorean influences in the later Roman Empire. This may be true, but if the spread of these numerals goes back that far, then it is more likely due to the influences of trade rather than of philosophy.

Alexandria remained the center of antique mathematics. Original work continued, though compilation and commentarization became more and more the prominent form of science. Many results of the ancient mathematicians and astronomers have been transmitted to us through the works of these compilers; and it is sometimes quite difficult to find out what they transcribed and what they discovered themselves. In trying to understand the gradual decline of Greek mathematics we must also take its technical side into account: the clumsy geometrical mode of expression with the consistent rejection of algebraic notation, which made any advance beyond the conic sections almost impossible. Algebra and computation were left to the despised Orientals, whose lore was covered by a veneer of Greek civilization. It is wrong, however, to believe that Alexandrian mathematics was purely "Greek" in the traditional Euclidean-Platonic sense; computational arithmetics and algebra of an Egyptian-Babylonian type was cultivated side by side with abstract geometrical demonstrations. We have only to think of Ptolemy,

Heron, and Diophantos to become convinced of this fact. The only tie between the many races and schools was the common use of Greek.

12. One of the earliest Alexandrian mathematicians of the Roman period was Nicomachos of Gerasa (A.D. 100) whose "Arithmetic Introduction" is the most complete exposition extant of Pythagorean arithmetic. It deals in great part with the same subjects as the arithmetical books of Euclid's "Elements," but where Euclid represents numbers by straight lines, Nicomachos uses arithmetical notation with ordinary language when undetermined numbers are expressed. His treatment of polygonal numbers and pyramidal numbers was of influence on medieval arithmetic, especially through Boetius.

One of the greatest documents of this second Alexandrian period was Ptolemy's "Great Collection," better known under the Arabicized title of "Almagest" (c. 150 A.D.). The "Almagest" was an astronomical opus of supreme mastership and originality, even if many of the ideas may have come from Hipparchos or Kidinnu and other Babylonian astronomers. It also contained a trigonometry, with a table of chords belonging to different angles, equivalent to a sine table of angles ranging from 0° to 90°, ascending by halves of an angle. Ptolemy found for the chord of 1° the value (1, 2, 50), $= \frac{1}{60} + \frac{2}{60^2} + \frac{50}{60^3} =$ .017268; the correct value is .017453; for π the value (3, 8, 30) = 3.14166. We find in the "Almagest" the formula for the sine and cosine of the sum and difference of two angles, together with a beginning of spherical trigonometry. The theorems were expressed in geometrical form—our present trigonometrical notation only dates from Euler in the Eighteenth Century. We also meet in this book "Ptolemy's theorem" for a quadrilateral inscribed in a circle. In Ptolemy's "Planisphaerium" we find a discussion of stereographic projection and of latitude and longitude in a sphere, which are ancient examples of coordinates.

Somewhat older than Ptolemy was Menelaos (c. 100 A.D.), whose "Sphaerica" contained a geometry of the sphere, with a discussion of spherical triangles, a subject which is missing in Euclid. Here we find "Menelaos' theorem" for the triangle in its extension to the sphere. Where Ptolemy's astronomy contained a good deal of computational work in sexagesimal fractions, Menelaos' treatise was geometrical in the pure Euclidean tradition.

To the period of Menelaos may also belong Heron, at any rate we know that he described accurately a lunar eclipse of 62 A.D[1]. Heron was an encyclopedic writer who wrote on geometrical, computational, and mechanical subjects; they show a curious blend of the Greek and Oriental. In his "Metrica" he derived the "Heronic" formula for the area of a triangle $A = \sqrt{s(s-a)(s-b)(s-c)}$ in purely geometrical form; the proposition itself has been ascribed to Archimedes. In the same "Metrica" we find typical Egyptian unit fractions, as in the approximation of $\sqrt{63}$ by $7 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$. Heron's formula for the volume of a frus-

[1]O. Neugebauer. *Über eine Methode zur Distanzbestimmung Alexandria—Rom bei Heron.* Hist. fil. Medd. Danske Vid. Sels. 26 (1938) No. 2, 28 pp.

tum of a square pyramid can readily be reduced to the one found in the ancient Moscow Papyrus. His measurement of the volume of the five regular polyhedra, on the contrary, was again in the spirit of Euclid.

13. The Oriental touch is even stronger in the "Arithmetica" of Diophantos (c. 250 A.D.). Only six of the original books survive; their total number is a matter of conjecture. Their skillful treatment of indeterminate equations shows that the ancient algebra of Babylon or perhaps India not only survived under the veneer of Greek civilization but also was improved by a few active men. How and when it was done is not known, just as we do not know who Diophantos was—he may have been a Hellenized Babylonian. His book is one of the most fascinating treatises preserved from Greek-Roman antiquity.

Diophantos' collection of problems is of wide variation and their solution is often highly ingenious. "Diophantine analysis" consists in finding answers to such indeterminate equations as $Ax^2 + Bx + C = y^2$, $Ax^3 + Bx^2 + Cx + D = y^2$, or sets of these equations. Typical of Diophantos is that he was only interested in positive rational solutions; he called irrational solutions "impossible" and was careful to select his coefficients so as to get the positive rational solution he was looking for. Among the equations we find $x^2 - 26y^2 = 1$, $x^2 - 30y^2 = 1$, now known as "Pell" equations. Diophantos also has several propositions in the theory of numbers, such as the theorem (III, 19) that if each of two integers is the sum of two squares their product can be resolved in two ways into two squares. There are

theorems about the division of a number into the sum of three and four squares.

In Diophantos we find the first systematic use of algebraic symbols. He has a special sign for the unknown, for the minus, for reciprocals. The signs are still of the nature of abbreviations rather than algebraic symbols in our sense (they form the so-called "rhetoric" algebra); for each power of the unknown there exists a special symbol[1]. There is no doubt that we have here not only, as in Babylon, arithmetical questions of a definite algebraic nature, but also a well developed algebraic notation which was greatly conducive to the solution of problems of greater complication than were ever taken up before.

14. The last of the great Alexandrian mathematical treatises was written by Pappos (end Third Century). His "Collection" ("Synagoge") was a kind of handbook to the study of Greek geometry with historical annotations, improvements, and alterations of existing theorems and demonstrations. It was to be read with the original works rather than independently. Many results of ancient authors are known only in the form in which Pappos preserved them. Examples are the problems dealing with the quadrature of the circle, the duplication of the cube, and the trisection of the angle.

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[1]Papyrus 620 of the University of Michigan, acquired in 1921, contains some problems in Greek algebra dating to a period before Diophantos, perhaps early second century A.D. Some symbols found in Diophantos appear in this manuscript. See F. E. Robbins, Classical Philology 24 (1929) pp. 321-329; K. Vogel, ib. 25 (1930) pp. 373-375.

Interesting is his chapter on isoperimetric figures, in which we find the circle has larger area than any regular polygon of equal perimeter. Here is also the remark that the cells in the honeycomb satisfy certain maximum-minimum properties.[1] Archimedes' semi-regular solids are also known through Pappos. Like Diophantos' "Arithmetica," the "Collection" is a challenging book whose problems inspired much further research in later days.

The Alexandrian school gradually died with the decline of antique society. It remained, as a whole, a bulwark of paganism against the progress of christianity, and several of its mathematicians have also left traces in the history of ancient philosophy. Proclos (410–485), whose "Commentary on the First Book of Euclid" is one of our main sources of the history of Greek mathematics, headed a Neoplatonist school in Athens. Another representative of this school, in Alexandria, was Hypatia, who wrote commentaries on the classical mathematicians. She was murdered in 415 by the followers of St. Cyril, a fate which inspired a novel by Charles Kingsley[2]. These philosophical schools with their commentators had their ups and downs for centuries. The Academy in Athens was discontinued as "pagan" by the Emperor Justinian (529), but by this time there were again schools in such places as Constantinople Jundīshāpūr. Many old codices survived in Constantinople, while commentators continued to perpetuate the memory of Greek science and philosophy in the Greek language. In 630 Alexandria was taken by the Arabs, who replaced the upper layer of Greek civilization in Egypt by an upper layer of Arabic. There is no reason to believe that the Arabs destroyed the famous Alexandrian Library, since it is doubtful whether this library still existed at that time. As a matter of fact, the Arabic conquests did not materially change the character of the mathematical studies in Egypt. There may have been a retrogression, but when we hear of Egyptian mathematics again it is still following the ancient Greek-Oriental tradition (e.g. Alhazen).

[1]A full discussion of this problem in D'Arcy W. Thompson, *On Growth and Form* (Cambridge, 2nd ed., 1942).

[2]See also Voltaire, *Dictionnaire Philosophique*, art. Hypatie (Oeuvres, ed., 1819, tome 36, p. 458).

15. We end this chapter with some remarks on Greek arithmetic and logistics. Greek mathematicians made a difference between "arithmetica" or science of numbers ("arithmoi") and "logistics" or practical computation. The term "arithmos" expressed only a natural number, a "quantity composed of units" (Euclid VII, Def. 2, this also meant that "one" was not considered a number). Our conception of real number was unknown. A line segment, therefore, had not always a length. Geometrical reasoning replaced our work with real numbers. When Euclid wanted to express that the area of a triangle is equal to half base times altitude, he had to state that it is half the area of a parallelogram of the same base and lying between the same parallels (Euclid I, 41). Pythagoras' theorem was a relation between the areas of three squares and not between the lengths of three sides. This conception must be considered as a deliberate act brought about by the victory of Platonic idealism among those sec-

tions of the Greek ruling class interested in mathematics, since the contemporary Oriental conceptions concerning the relation of algebra and geometry did not admit any restriction of the number concept. There is every reason to believe that, for the Babylonians, Pythagoras' theorem was a numerical relation between the lengths of sides, and it was this type of mathematics with which the Ionian mathematicians had become acquainted.

Ordinary computational mathematics known as "logistics" remained very much alive during all periods of Greek history. Euclid rejected it, but Archimedes and Heron used it with ease and without scruples. Actually it was based on a system of numeration which changed with the times. The early Greek method of numeration was based on an additive decimal principle like that of the Egyptians and the Romans. In Alexandrian times, perhaps earlier, a method of writing numbers appeared which was used for fifteen centuries, not only by scientists but also by merchants and administrators. It used the successive symbols of the Greek alphabet to express, first our symbols 1, 2, ······, 9, then the tens from 10 to 90, and finally the hundreds from 100 to 900 ($\alpha = 1, \beta = 2$, etc.) Three extra archaic letters were added to the 24 letters of the Greek alphabet in order to obtain the necessary 27 symbols. With the aid of this system every number less than 1000 could be written with at most three symbols, e.g. 14 as $\iota\delta$, since $\iota = 10$, $\delta = 4$; numbers larger than 1000 could be expressed by a simple extension of the system. It is used in the existing manuscripts of Archimedes, Heron,

and all the other classical authors. There is archaeological proof that it was taught in the schools.

This was a decimal non-position system, both $\iota\delta$ and $\delta\iota$ could only mean 14. This lack of place value and the use of no less than 27 symbols have occasionally been taken as a proof of the inferiority of the system. The ease with which the ancient mathematicians used it, its acceptance by Greek merchants even in rather complicated transactions, its long persistence—in the East Roman Empire until its very end in 1453—seem to point to certain advantages. Some practice with the system can indeed convince us that it is possible to perform the four elementary operations easily enough once the meaning of the symbols is mastered. Fractional calculus with a proper notation is also simple; but the Greeks were inconsistent because of their lack of a uniform system. They used Egyptian unit fractions, Babylonian sexagesimal fractions, and also fractions in a notation reminiscent of ours. Decimal fractions were never introduced, but this great improvement appears only late in the European Renaissance after the computational apparatus had extended far beyond anything ever used in antiquity; even then decimal fractions were not adopted in many schoolbooks until the Eighteenth and Nineteenth Century.

It has been argued that this alphabetical system has been detrimental to the growth of Greek algebra, since the use of letters for definite numbers prevented their use for denoting numbers in general, as we do in our algebra. Such a formal explanation of the absence of a Greek algebra before Diophantos must be rejected, even

if we accept the great value of an appropriate notation. If the classical authors had been interested in algebra they would have created the appropriate symbolism. with which Diophantos actually made a beginning. The problem of Greek algebra can be elucidated only by further study of the connections between Greek mathematicians and Babylonian algebra in the framework of the entire relationship of Greece and the Orient.

Literature.

The Greek classical authors are available in excellent texts with the principal texts also available in English translations. The best introduction is offered in the following books:

T. L. Heath, *A History of Greek Mathematics* (2 vols., Cambridge, 1921).

T. L. Heath, *A Manual of Greek Mathematics* (Oxford, 1931).

T. L. Heath, *The Thirteen Books of Euclid's Elements* (3 vols., Cambridge, 1908).

See also:

P. Ver Eecke, *Oeuvres complètes d'Archimède* (Bruxelles, 1921).

P. Ver Eecke, *Pappus d'Alexandrie. La Collection mathématique.* (Paris, Bruges, 1933).

P. Ver Eecke, *Proclus de Lycie. Les Commentaires sur le Premier Livre des Elèments d'Euclide* (Bruges, 1948).

G. Loria, *Le scienze esatte nell' antica Grecia* (Milan, 2nd ed, 1914).

G. J. Allman, *Greek Geometry from Thales to Euclid* (Dublin, 1889).

J. Gow, *A Short History of Greek Mathematics* (Cambridge, 1844. Reprinted 1951).

K. Reidemeister, *Die Arithmetik der Griechen*, Hamburg Math. Seminar Einzelschriften 26 (1939).

Comparative Greek, Latin, and English texts in:

I. Thomas, *Selections Illustrating the History of Greek Mathematics* (Cambridge, Mass., London, 1939).

Further text criticism in:

P. Tannery, *Pour l'histoire de la science hellène* (2nd.ed. Paris, 1930).

P. Tannery, *Mémoires scientifiques* (vols. 1-4).

H. Vogt, *Die Entdeckungsgeschichte des Irrationalen nach Plato und anderen Quellen des 4ten Jahrhunderts*, Bibliotheca mathematica (3) 10 (1909-10) pp. 97-105.

E. Sachs, *Die fünf Platonischen Körper* (Berlin, 1917).

E. Frank, *Plato und die sogenannten Pythagoreer* (Halle, 1923).

S. Luria, *Die Infinitesimaltheorie der antiken Atomisten*, Quellen und Studien 2 (1932) pp. 106-185.

A good and critical survey of the different hypotheses concerning Greek mathematics in:

E. Dijksterhuis, *De elementen van Euclides* (2 vols., Groningen, 1930, in the Dutch language).

On Zeno's paradoxes see, apart from Van der Waerden, l. c. p. 50:

F. Cajori, *The History of Zeno's Arguments on Motion,* Am. Math. Monthly 22 (1915), 8 articles (See also Isis 3).

On the relation of Greek to Oriental astronomy:

O. Neugebauer, *The History of Ancient Astronomy. Problems and Methods,* Journ. of Near Eastern Studies 4 (1945) pp. 1-38.

See further:

M. R. Cohen–I. E. Drabkin, *A Source Book in Greek Science* (London, 1948).

T. L. Heath, *Mathematics in Aristotle* (Oxford, 1949).

B. L. Van der Waerden, *Ontwakende Wetenschap* (Groningen, 1950).

This book, in Dutch, deals with Egyptian, Babylonian, and Greek mathematics.

CHAPTER IV

The Orient after the Decline of Greek Society

1. The ancient civilization of the Near East never disappeared despite all Hellenistic influence. Both Oriental and Greek influences are clearly revealed in the science of Alexandria; Constantinople and India were also important meeting grounds of East and West. In 395 Theodosios I founded the Byzantine Empire; its capital Constantinople was Greek, but it was the center of administration of vast territories where the Greeks were only a section of the urban population. For a thousand years this empire fought against the forces from the East, North, and West, serving at the same time as a guardian of Greek culture and as a bridge between the Orient and the Occident. Mesopotamia became independent of the Romans and Greeks as early as the second century A.D., first under the Parthian kings, later (266) under the purely Persian dynasty of the Sassanians. The Indus region had for some centuries several Greek dynasties, which disappeared by the first century A.D.; but the native Indian kingdoms which followed kept up cultural relations with Persia and the West.

The political hegemony of the Greeks over the Near East disappeared almost entirely with the sudden growth of Islam. After 622, the year of the Hegira, the Arabs conquered large sections of Western Asia in an amazing sweep and before the end of the seventh century had occupied parts of the West Roman empire

as far as Sicily, North Africa, and Spain. Wherever they went they tried to replace the Greek-Roman civilization by that of the Islam. The official language became Arabic, instead of Greek or Latin; but the fact that a new language was used for the scientific documents tends to obscure the truth that under Arabic rule a considerable continuity of culture remained. The ancient native civilizations had even a better chance to survive under this rule than under the alien rule of the Greeks. Persia, for instance, remained very much the ancient country of the Sassanians, despite the Arabic administration. However, the contest between the different traditions continued, only now in a new form. Throughout the whole period of Islamic rule there existed a definite Greek tradition holding its own against the different native cultures.

2. We have seen that the most glorious mathematical results of this competition and blending of Oriental and Greek culture during the heydays of the Roman Empire appeared in Egypt. With the decline of the Roman Empire the center of mathematical research began to shift to India and later back to Mesopotamia. The first well-preserved Indian contributions to the exact sciences are the "Siddhāntās," of which one, the "Sūrya," may be extant in a form resembling the original one (c. 300-400 A.D.). These books deal mainly with astronomy and operate with epicycles and sexagesimal fractions. These facts suggest influence of Greek astronomy, perhaps transmitted in a period antedating the "Almagest"; they also may indicate direct contact with Babylonian astronomy. In addition the "Siddhāntās" show

many native Indian characteristics. The "Sūrya Siddhāntā" has tables of sines (jyā) instead of chords.

The results of the "Siddhāntās" were systematically explained and extended by schools of Indian mathematicians, mainly centered in Ujjain (Central India) and Mysore (S. India). From the Fifth Century A.D. on, names and books of individual Indian mathematicians have been preserved; some books are even available in English translations.

The best known of these mathematicians are Āryabhata (called "the first," c. 500) and Brahmagupta (c. 625). The whole question of their indebtedness to Greece, Babylon, and China is a subject of much conjecture; but they show at the same time considerable originality. Characteristic of their work are the arithmetical-algebraic parts, which bear in their love for indeterminate equations some kinship to Diophantos. These writers were followed in the next centuries by others working in the same general field; their works were partly astronomical, partly arithmetic-algebraical, and had excursions into mensuration and trigonometry. Āryabhata I had for π the value 3.1416. A favorite subject was the finding of rational triangles and quadrilaterals, in which Mahāvirā of the Mysore school (c. 850) was particularly prolific. Around 1150 we find in Ujjain, where Brahmagupta had worked, another excellent mathematician, Bhāskara. The first general solution of indeterminate equations of the first degree $ax + by = c$ (a, b, c integers) is found in Brahmagupta. It is therefore, strictly speaking, incorrect to call linear indeterminate equations Diophantine equations. Where Diophantos still accepted fractional solutions, the Hin-

dus were only satisfied with integer solutions. They also advanced beyond Diophantos in admitting negative roots of equations, though this may again have been an older practice suggested by Babylonian astronomy. Bhāskara, for instance, solved $x^2 - 45x = 250$ by $x = 50$ and $x = -5$; he indulged in some scepticism as to the validity of the negative root. His "Lilāvati" was for many centuries a standard work on arithmetic and mensuration in the East; the emperor Akbar had it translated into Persian (1587). In 1832 an edition was published in Calcutta[1].

3. The best known achievement of Hindu mathematics is our present decimal position system. The decimal system is very ancient, and so is the position system; but their combination seems to have originated in India, where in the course of time it was gradually imposed upon older non-position systems. Its first known occurrence is on a plate of the year 595 A.D., where the date 346 is written in decimal place value notation. The Hindus long before this epigraphic record had a system of expressing large numbers by means of words arranged according to a place value

[1]Brahmagupta states somewhere in his book that some of his problems were proposed "simply for pleasure". This confirms that mathematics in the Orient had long since evolved from its purely utilitarian function. One hundred and fifty years later Alcuin, in the West, wrote his "Problems for the Quickening of the Mind of the Young", expressing a similar non-utilitarian purpose. Mathematics in the form of the intellectual puzzle has often contributed essentially to the progress of science by opening new fields. Some puzzles still await their integration into the main body of mathematics.

method. There are early texts in which the word "Sūnya," meaning zero, is explicitly used[1]. The so-called Bakshāli manuscript, consisting of seventy leaves of birch bark of uncertain origin and date (estimates range from the Third to the Twelfth Century A.D.), and with traditional Hindu material on indeterminate and quadratic equations as well as approximations, has a dot to express zero. The oldest epigraphic record with a sign for zero dates from the Ninth Century. This is all much later than the occurrence of a sign for zero in Babylonian texts.

The decimal place value system slowly penetrated along the caravan roads into many parts of the Near East, taking its place beside other systems. Penetration into Persia, perhaps also Egypt, may very well have happened in the Sassanian period (224-641), when the contact between Persia, Egypt, and India was close. In this period the memory of the ancient Babylonian place value system may still have been alive in Mesopotamia. The oldest definite reference to the Hindu place value system outside of India is found in a work of 662 written by Severus Sēbōkht, a Syrian bishop. With Al-Fazārī's translation of the "Siddhāntās" into Arabic (c. 773) the Islamic scientific world began to be acquainted with the so-called Hindu system. This system began to be more widely used in the Arabic world and beyond, though the Greek system of numeration also remained in use as well as other native systems.

[1]This may be compared to the use of the concept of the "void" (kenos) in Aristotle's "Physica" IV. 8.215[b]. See C. B. Boyer. *Zero: the symbol, the concept, the number.* Nat. Mathem. Magazine 18 (1944) pp. 323-330.

Social factors may have played a role—the Oriental tradition favoring the decimal place value method against the method of the Greeks. The symbols used to express the place value numerals show wide variations but there are two main types: the Hindu symbols used by the Eastern Arabs; and the so-called "ġobâr" (or ghubār) numerals used in Spain among the Western Arabs. The first symbols are still used in the Arab world but our present numeral system seems to be derived from the "ġobâr" system. There exists an already mentioned theory of Woepcke, according to which the "ġobâr" numerals were in use in Spain when the Arabs arrived, having reached the West through the Neo-Pythagoreans of Alexandria as early as 450 A.D.[1]

4. Mesopotamia, which under the Hellenistic and Roman rules had become an outpost of the Roman empire, reconquered its central position along the trade routes under the Sassanians, who reigned as native Persian kings over Persia in the tradition of Cyrus and Xerxes. Little is known about this period in Persian history, especially about its science, but the legendary history—the Thousand and One Nights, Omar Khayyam, Firdawsī—confirms the meagre historical record that the Sassanian period was an era of cultural splendor. Situated between Constantinople, Alexandria,

[1]Comp. S. Gandz, *The Origin of the Ghubār Numerals*, Isis 16 (1931) pp. 393-424. There exists also a theory of N. Bubnov, which holds that the ġobâr forms were derived from ancient Roman-Greek symbols used on the abacus. See also the footnote in F. Cajori, *History of Mathematics* (New York, 1938) p. 90, as well as Smith-Karpinski, (quoted p. 97) p. 71.

India, and China, Sassanian Persia was a country where many cultures met. Babylon had disappeared but was replaced by Seleucia-Ktesiphon, which again made place for Bagdad after the Arabic conquest of 641. This conquest left much of ancient Persia unaffected, though Arabic replaced Pehlevi as the official language. Even Islām was only accepted in a modified form (Shīʿism); Christians, Jews, and Zoroastrians continued to contribute to the cultural life of the Bagdad caliphate.

The mathematics of the Islamic period shows the same blend of influences with which we have become familiar in Alexandria and in India[1]. The Abbāsid caliphs, notably Al-Mansūr (754-775), Hārūn al-Rashīd (786-809), and Al-Ma'-mūn (813-833) promoted astronomy and mathematics, Al-Ma'-mūn even organizing at Bagdad a "House of Wisdom" with a library and an observatory. Islamic activities in the exact sciences, which began with Al-Fāzarī's translation of the "Siddhāntās," reached its first height with a native from Khiva, Muhammad ibn Mūsā al-Khwārizmī, who flourished about 825. Muhammad wrote several books on mathematics and astronomy. His arithmetic explained the Hindu system of numeration. Although lost in the original Arabic, a Latin translation of the Twelfth Century is extant. This book was one of the means by which Western Europe became acquainted with the decimal position. The title of the translation,

[1]All accounts of "Arabic" mathematics must remain tedious repetition of second hand and third hand information as long as only some works such as Al-Khwārizmī and Khayyam are available in translation. A history of "Arabic" mathematics by a competent "Arabic" scholar does not exist. Suter's book was a mere beginning.

Courtesy of The Metropolitan Museum of Art

TOMB OF OMAR KHAYYAM IN NISHAPUR

"Algorithmi de numero Indorum," added the term "algorithmus"—a latinization of the author's name—to our mathematical language. Something similar happened to Muhammad's algebra, which had the title "Hisab al-jabr wal-muqābala" (lit. "science of reduction and cancellation", probably meaning "science of equations"). This algebra, of which the Arabic text is extant, also became known in the West through Latin translations, and they made the word "al-jabr" synonymous with the whole science of "algebra", which, indeed, until the middle of the Nineteenth Century was nothing but the science of equations.

This "algebra" contains a discussion of linear and quadratic equations, but without any algebraic formalism. Even the Diophantine "rhetoric" symbolism was absent. Among these equations are the three types characterized by $x^2 + 10x = 39$, $x^2 + 21 = 10x$, $3x + 4 = x^2$, which had to be separately treated as long as positive coefficients were the only ones which were admitted. These three types reappear frequently in later texts—"thus the equation $x^2 + 10x = 39$ runs like a thread of gold through the algebras for several centuries," writes Professor L. C. Karpinski. Much of the reasoning is geometric. Muhammad's astronomical and trigonometrical tables (with sines and tangents) also belong to the Arabic works which later were translated into Latin. His geometry is a simple catalogue of mensuration rules; it is of some importance because it can be directly traced to a Jewish text of 150 A.D. It shows a definite lack of sympathy with the Euclidean tradition. Al-Khwārizmī's astronomy was an abstract of the "Siddhāntās," and therefore may show some indirect

Greek influence by way of a Sanskrit text. The works of Al-Khwārizmī as a whole seem to show Oriental rather than Greek influence[1], and this may have been deliberate.

Al-Khwārizmī's work plays an important role in the history of mathematics, for it is one of the main sources through which Indian numerals and Arabic algebra came to Western Europe. Algebra, until the middle of the Nineteenth Century, revealed its Oriental origin by its lack of an axiomatic foundation, in this respect sharply contrasting with euclidean geometry. The present day school algebra and geometry still preserve these tokens of their different origin.

5. The Greek tradition was cultivated by a school of Arabic scholars who faithfully translated the Greek classics into Arabic—Apollonius, Archimedes, Euclid, Ptolemy, and others. The general acceptance of the name "Almagest" for Ptolemy's "Great Collection" shows the influence of the Arabic translations upon the West. This copying and translating has preserved many a Greek classic which otherwise would have been lost. There was a natural tendency to stress the computational and practical side of Greek mathematics at the cost of its theoretical side. Arabic astronomy was particularly interested in trigonometry—the word "sinus" is a Latin translation of the Arabic spelling of the Sanscrit jyā. The sines correspond to half the chord of the double arc (Ptolemy used the whole chord), and were conceived as lines, not as numbers. We find a

[1]S. Gandz, *The Sources of Al-Khwārizmī's Algebra*, Osiris 1 (1936) pp. 263-277.

good deal of trigonometry in the works of Al-Battānī (Albategnius, bef. 858–929), one of the great Arabic astronomers who had a table of cotangents for every degree ("umbra extensa") as well as the cosine rule for the spherical triangle.

This work by Al-Battānī shows us that Arabic writers not only copied but also contributed new results through their mastery of both Greek and Oriental methods. Abūl-l-Wafā' (940–997/8) derived the sine theorem of spherical trigonometry, computed sine tables for intervals of 15′ of which values were correct in eight decimal places, introduced the equivalents of secant and cosecant, and played with geometrical constructions using a compass of one fixed opening. He also continued the Greek study of cubic and biquadratic equations. Al-Karkhī (beginning of the Eleventh Century), who wrote an elaborate algebra following Diophantos, had interesting material on surds, such as the formulas $\sqrt{8} + \sqrt{18} = \sqrt{50}$, $\sqrt[3]{54} - \sqrt[3]{2} = \sqrt[3]{16}$. He showed a definite tendency in favor of the Greeks; his "neglect of Hindu mathematics was such that it must have been systematic"[1].

6. We need not follow the many political and ethnological changes in the world of Islām. They brought ups and downs in the cultivation of astronomy and mathematics; certain centers disappeared, others flourished for a while; but the general character of the Islamic type of science remained virtually unchanged. We shall mention only a few highlights.

About 1000 B.C. new rulers appeared in Northern

[1]G. Sarton, *Introduction to the History of Science* I, p. 719.

Persia, the Saljūq (Selchuk) Turks, whose empire flourished around the irrigation center of Merv. Here lived Omar Khayyam (c. 1038/48-1123/24), known in the West as the author of the "Rubaiyat" (in the Fitzgerald translation, 1859); he was an astrononer and a philosopher:

> "Ah, but my Computations, People say,
> Have squared the Year to human Compass, eh?
> If so, by striking from the Calendar
> Unborn tomorrow, and dead Yesterday". (LIX)

Here Omar may have referred to his reform of the old Persian calendar, which instituted an error of one day in 5000 years (1540 or 3770 years according to different interpretations), whereas our present Gregorian calendar has an error of one day in 3330 years. His reform was introduced in 1079 but was later replaced by the Muslim lunar calendar. Omar wrote an "Algebra," which represented a considerable achievement, since it contained a systematic investigation of cubic equations. Using a method occasionally used by the Greeks, he determined the roots of these equations as intersection of two conic sections. He had no numerical solutions and discriminated—also in Greek style—between "geometrical" and "arithmetical" solutions, the latter existing only if the roots are positive rational. This approach was therefore entirely different from that of the Sixteenth Century Bolognese mathematicians, who used purely algebraical methods.

After the sack of Bagdad in 1256 by the Mongols a new center of learning sprang up near the same place at the observatory of Maragha, built by the Mongol ruler Hulagu for Nasīr al-dīn (Nasir-eddin, 1201-1274).

Here again arose an institute where the whole of Oriental science could be pooled and matched with the Greek. Naṣir separated trigonometry as a special science from astronomy; his attempts to "prove" Euclid's parallel axiom show that he appreciated the theoretical approach of the Greeks. Naṣir's influence was widely felt later in Renaissance Europe; as late as 1651 and 1663 John Wallis used Naṣir's work on the Euclidean postulate.

An important figure in Egypt was Ibn Al-Haitham (Alhazen, c. 965-1039), the greatest Muslim physicist, whose "Optics" had a great influence on the West. He solved the "problem of Alhazen," in which we are asked to draw from two points on the plane of a circle lines meeting at the point of the circumference and making equal angles with the normal at that point. This problem leads to a biquadratic equation and was solved in the Greek way by a hyperbola meeting a circle. Alhazen also used the exhaustion method to compute the volumes of figures obtained by revolving a parabola about any diameter or ordinate. One hundred years before Alhazen there lived in Egypt the algebrist Abū Kāmil, who followed and extended the work of Al-Khwārizmī. He influenced not only Al-Karkhī, but also Leonardo of Pisa.

Another center of learning existed in Spain. One of the important astronomers at Cardoba was Al-Zarqāli, (Arzachel, c. 1029-c. 1087), the best observer of his time and the editor of the so-called Toledan planetary tables. The trigonometrical tables of this work, which was translated into Latin, had some influence on the development of trigonometry in the Renaissance.

In China we must mention Tsu Ch'ung Chi (430-501) who found for π the value 355/113; the mathematicians and astrononers of the T'ang dynasty (610-907) who compiled the "Ten Classics"; and Ch'in Chiu-Shao (Thirteenth Century) who gave numerical solutions of equations of higher degree. Ch'in's solution of the equation:

$$x^4 - 763200\, x^2 + 40642560000 = 0$$

followed a method similar to that which we now know as Horner's.

Some of these achievements are not without interest, but they show no essential progress beyond the limits set by the Greeks and the Babylonians. The same holds for Japanese mathematics, about which information begins to reach us from the Twelfth Century on. All this work in mathematics remained at best semi-stagnating. We lose interest in it when in the Sixteenth Century a new type of mathematics began to flourish in the West.[1]

Literature.

(See also at the end of Chapter II.)

H. Suter, *Die Mathematiker und Astronomer der Araber und ihre Werke* (Leipzig, 1900, Nachträge, 1902).

[1]In the Sixteenth and Seventeenth Centuries Chinese and Japanese mathematics begins to become more interesting. This was partly due to contact with Europe. Western mathematics and astronomy were introduced into China by Father Matteo Ricci, who stayed in Peking from 1583 until his death in 1610. See H. Bosmans, *L'oeuvre scientifique de Mathieu Ricci, S.J.*, Revue des Questions Scient. (Jan. 1921) 16 pp.

See H. P. J. Renaud, Isis 18 (1932) pp. 166–83.

D. S. Kasir, *The Algebra of Omar Khayyam* (New York, 1931) p. 126.

B. Datta, *The Science of the Sulba, a Study in Early Hindu Geometry* (London, 1932).

D. E. Smith—L. C. Karpinski, *The Hindu-Arabic Numerals* (Boston, 1911).

L. C. Karpinski, *Robert of Chester's Latin Translation of the Algebra of Al-Khwarizmi* (New York, 1915).

T. Hayashi, *A Brief History of the Japanese Mathematics*, Nieuw Archief voor Wiskunde 2° ser., 6 (1904-05) pp. 296-361.

D. E. Smith, *Unsettled Questions Concerning the Mathematics of China*, Scient. Monthly 33 (1931) pp. 244-250.

H. T. Colebrooke, *Algebra, with Arithmetic and Mensuration from the Sanskrit of Brahmagupta and Bhāscara* (London, 1817, revised by H. C. Banerji, Calcutta, 2nd. ed. 1927).

F. Rosen, *The Algebra of Mohammed ben Musa* (London, 1831).

See S. Gandz, Quellen und Studien 2A (1932) pp. 61–85.

W.E.Clark, *The Āryabhatya of Āryabhata* (Chicago, 1930).

CHAPTER V

The Beginnings in Western Europe

1. The most advanced section of the Roman Empire from both an economic and a cultural point of view had always been the East. The Western part had never been based on an irrigation economy; its agriculture was of the extensive kind which did not stimulate the study of astronomy. Actually the West managed very well in its own way with a minimum of astronomy, some practical arithmetics, and some mensuration for commerce and surveying; but the stimulus to promote these sciences came from the East. When East and West separated politically this stimulation almost disappeared. The static civilization of the Western Roman Empire continued with little interruption or variation for many centuries; the Mediterranean unity of antique civilization also remained unchanged—and was not even very much affected by the barbaric conquests. In all Germanic kingdoms, except perhaps those of Britain, the economic conditions, the social institutions, and the intellectual life remained fundamentally what they had been in the declining Roman Empire. The basis of economic life was agriculture, with slaves gradually replaced by free and tenant farmers; but in addition there were prosperous cities and a large-scale commerce with a money economy. The central authority in the Greek-Roman world after the fall of the Western Empire in 476 was shared by the emperor in Constantinople and the popes of Rome. The Catholic Church of the

West through its institutions and language continued as best it could the cultural tradition of the Roman Empire among Germanic kingdoms. Monasteries and cultured laymen kept some of the Greek-Roman civilization alive.

One of these laymen, the diplomat and philosopher Anicius Manilius Severinus Boetius, wrote mathematical texts which were considered authoritative in the Western world for more than a thousand years. They reflect the cultural conditions, for they are poor in content and their very survival may have been influenced by the belief that the author died in 524 as a martyr to the Catholic faith. His "Institutio arithmetica," a superficial translation of Nicomachus, did provide some Pythagorean number theory which was absorbed in medieval instruction as part of the ancient trivium and quadrivium: arithmetic, geometry, astronomy, and music.

It is difficult to establish the period in the West in which the economy of the ancient Roman Empire disappeared to make room for a new feudal order. Some light on this question is shed by the hypothesis of H. Pirenne[1], according to which the end of the ancient Western world came with the expansion of Islām. The Arabs dispossessed the Byzantine Empire of all its provinces on the Eastern and Southern shores of the Mediterranean and made the Eastern Mediterranean a closed Muslim lake. They made commercial relations between the Near Orient and the Christian Occident extremely difficult for several centuries. The intellectual avenue between the Arabic world and the Northern

[1]H. Pirenne, *Mahomet and Charlemagne* (London, 1939).

parts of the former Roman Empire, though never wholly closed, was obstructed for centuries.

Then in Frankish Gaul and other former parts of the Roman Empire large-scale economy subsequently vanished; decadence overtook the cities; returns from tolls became insignificant. Money economy was replaced by barter and local marketing. Western Europe was reduced to a state of semi-barbarism. The landed aristocracy rose in significance with the decline of commerce; the North Frankish landlords, headed by the Carolingians, became the ruling power in the land of the Franks. The economic and cultural center moved to the North, to Northern France and Britain. The separation of East and West limited the effective authority of the pope to the extent that the papacy allied itself with the Carolingians, a move symbolized by the crowning of Charlemagne in 800 as Emperor of the Holy Roman Empire. Western society became feudal and ecclesiastical, its orientation Northern and Germanic.

2. During the early centuries of Western feudalism we find little appreciation of mathematics even in the monasteries. In the again primitive agricultural society of this period the factors stimulating mathematics, even of a directly practical kind, were nearly nonexistent; and monasteric mathematics was no more than some ecclesiastical arithmetic used mainly for the computation of Easter-time (the so-called "computus"). Boetius was the highest source of authority. Of some importance among these ecclesiastical mathematicians was the British born Alcuin, associated with the court of Charlemagne, whose Latin "Problems for the Quickening

of the Mind" (see p. 86) contained a selection which have influenced the writers of text books for many centuries. Many of these problems date back to the ancient Orient. For example:

"A dog chasing a rabbit, which has a start of 150 feet, jumps 9 feet every time the rabbit jumps 7. In how many leaps does the dog overtake the rabbit?"

"A wolf, a goat, and a cabbage must be moved across a river in a boat holding only one beside the ferry man. How must he carry them across so that the goat shall not eat the cabbage, nor the wolf the goat?"

Another ecclesiastical mathematician was Gerbert, a French monk, who in 999 became pope under the name of Sylvester II. He wrote some treatises under the influence of Boetius, but his chief importance as a mathematician lies in the fact that he was one of the first Western scholars who went to Spain and made studies of the mathematics of the Arabic world.

3. There are significant differences between the development of Western, of early Greek, and of Oriental feudalism. The extensive character of Western agriculture made a vast system of bureaucratic administrators superfluous, so that it could not supply a basis for an eventual Oriental despotism. There was no possibility in the West of obtaining vast supplies of slaves. When villages in Western Europe grew into towns these towns developed into self-governing units, in which the burghers were unable to establish a life of leisure based on slavery. This is one of the main reasons why the development of the Greek polis and the Western city, which during the early stages had much in common,

deviated sharply in later periods. The medieval townspeople had to rely on their own inventive genius to improve their standard of living. Fighting a bitter struggle against the feudal landlords—and with much civil strife in addition—they emerged victorious in the Twelfth, Thirteenth, and Fourteenth Centuries. This triumph was based not only on a rapid expansion of trade and money economy but also on a gradual improvement in technology. The feudal princes often supported the cities in their fight against the smaller landlords, and then eventually extended their rule over the cities. This finally led to the emergence of the first national states in Western Europe.

The cities began to establish commercial relations with the Orient, which was still the center of civilization. Sometimes these relations were established in a peaceful way, sometimes by violent means as in the many Crusades. First to establish mercantile relations were the Italian cities; they were followed by those of France and Central Europe. Scholars followed, or sometimes preceded, the merchant and the soldier. Spain and Sicily were the nearest points of contact between East and West, and there Western merchants and students became acquainted with Islamic civilization. When in 1085 Toledo was taken from the Moors by the Christians, Western students flocked to this city to learn the science of the Arabs. They often employed Jewish interpreters to converse and to translate, and so we find in Twelfth Century Spain Plato of Tivoli, Gherardo of Cremona, Adelard of Bath, and Robert of Chester, translating Arabic mathematical manuscripts into Latin. Thus Europe became familiar with Greek classics

through the Arabic; and by this time Western Europe was advanced enough to appreciate this knowledge.

4. As we have said, the first powerful commercial cities arose in Italy, where during the Twelfth and Thirteenth Centuries Genoa, Pisa, Venice, Milan, and Florence carried on a flourishing trade between the Arabic world and the North. Italian merchants visited the Orient and studied its civilization; Marco Polo's travels show the intrepidity of these adventurers. Like the Ionian merchants of almost two thousand years before, they tried to study the science and the arts of the older civilization not only to reproduce them, but also to put them to use in their own new and experimental system of merchant capitalism. The first of these merchants whose mathematical studies show a certain maturity was Leonardo of Pisa.

Leonardo, also called Fibonacci ("son of Bonaccio"), travelled in the Orient as a merchant. On his return he wrote his "Liber Abaci" (1202), filled with arithmetical and algebraical information which he had collected on his travels. In the "Practica Geometriae" (1220) Leonardo described in a similar way whatever he had discovered in geometry and trigonometry. He may have been an original investigator as well, since his books contain many examples which seem to have no exact duplicates in Arabic literature[1]. However he does quote Al-Khwārizmī, as for instance in the discussion of the equation $x^2 + 10x = 39$. The problem which leads to

[1] L. C. Karpinski, Amer. Math. Monthly 21 (1914) pp. 37-48, using the Paris manuscript of Abū Kāmil's algebra, claims that Leonardo followed Abū Kāmil in a whole series of problems.

the "series of Fibonacci" 0,1,1,2,3,5,8,13,21, ·· , of which each term is the sum of the two preceding terms, seems to be new and also his remarkably mature proof that the roots of the equation $x^3 + 2x^2 + 10x = 20$ cannot be expressed by means of Euclidean irrationalities $\sqrt{a + \sqrt{b}}$ (hence cannot be constructed by means of compasses and ruler only). Leonardo proved it by checking upon each of Euclid's fifteen cases, and then solved the positive root of this equation approximately, finding six sexagesimal places.

The series of Fibonacci resulted from the problem:

> How many pairs of rabbits can be produced from a single pair in a year if (a) each pair begets a new pair every month, which from the second month on becomes productive, (b) deaths do not occur?

The "Liber Abaci" is one of the means by which the Hindu-Arabic system of numeration was introduced into Western Europe. Their occasional use dates back to centuries before Leonardo, when they were imported by merchants, ambassadors, scholars, pilgrims, and soldiers coming from Spain and from the Levant. The oldest dated European manuscript containing the numerals is the "Codex Vigilanus," written in Spain in 976. However, the introduction of the ten symbols into Western Europe was slow; the earliest French manuscript in which they are found dates from 1275. The Greek system of numeration remained in vogue along the Adriatic for many centuries. Computation was often performed on the ancient abacus, a board with counters or pebbles (often simply consisting of lines drawn in sand) similar in principle to the counting

boards still used by the Russians, Chinese, Japanese, and by children on their baby-pens. Roman numerals were used to registrate the result of a computation on the abacus. Throughout the Middle Ages and even later we find Roman numerals in merchant's ledgers, which indicates that the abacus was used in the offices. The introduction of Hindu-Arabic numerals met with opposition from the public, since the use of these symbols made merchant's books difficult to read. In the statutes of the "Arte del Cambio" of 1299 the bankers of Florence were forbidden to use Arabic numerals and were obliged to use cursive Roman ones. Sometime during the Fourteenth Century Italian merchants began to use some Arabic figures in their account books.[1]

5. With the extension of trade interest in mathematics spread slowly to the Northern cities. It was at first mainly a practical interest, and for several centuries arithmetic and algebra were taught outside the universities by self-made reckon masters, usually ignorant

[1] In the Medici account books (dating from 1406) of the Selfridge collection on deposit at the Harvard Graduate School of Business Administration, Hindu-Arabic numerals frequently appear in the narrative or descriptive column. From 1439 onward they replace Roman numerals in the money or effective column of the books of primary entry: journals, wastebooks, etc., but not until 1482 were Roman numerals abandoned in the money column of the business ledgers of all but one Medici merchant. From 1494, only Hindu-Arabic numerals are used in all the Medici account books. (From a letter by Dr. Florence Edler De Roover.) See also, F. Edler, *Glossary of Medieval Terms of Business* (Cambridge, Mass., 1934) p. 389.

of the classics, who also taught bookkeeping and navigation. For a long time this type of mathematics kept definite traces of its Arabic origin, as words such as "algebra" and "algorithm" testify.

Speculative mathematics did not entirely die during the Middle Ages, though it was cultivated not among the men of practice, but among the scholastic philosophers. Here the study of Plato and Aristotle, combined with meditations on the nature of the Deity, led to subtle speculations on the nature of motion, of the continuum and of infinity. Origen had followed Aristotle in denying the existence of the actually infinite, but St. Augustine in the "Civitas Dei" had accepted the whole sequence of integers as an actual infinity. His words were so well chosen that Georg Cantor has remarked that the transfinitum cannot be more energetically desired and cannot be more perfectly determined and defended than was done by St. Augustine[1]. The scholastic writers of the Middle Ages, especially St. Thomas Aquinas, accepted Aristotle's "infinitum actu non datur,"[2] but considered every continuum as potentially divisible ad infinitum. Thus there was no smallest line, since every part of the line had the properties of the line. A point, therefore, was not a part of a line, because it was indivisible: "ex indivisilibus non

[1]G. Cantor, *Letter to Eulenburg* (1886), Ges. Abhandlungen (Berlin, 1932) pp. 401-402. The passage quoted by Cantor, Ch. XVIII of Book XII of "The City of God" (in the Healey translation) is entitled "Against such as say that things infinite are above God's knowledge."

[2]"There is no actually infinite."

potest compari aliquod continuum"[1]. A point could generate a line by motion. Such speculations had their influence on the inventors of the infinitesimal calculus in the Seventeenth Century and on the philosophers of the transfinite in the Nineteenth; Cavalieri, Tacquet, Bolzano and Cantor knew the scholastic authors and pondered over the meaning of their ideas.

These churchmen occasionally reached results of more immediate mathematical interest. Thomas Bradwardine, who became Archbishop of Canterbury, investigated star polygons after studying Boetius. The most important of these medieval clerical mathematicians was Nicole Oresme, Bishop of Lisieux in Normandy, who played with fractional powers. Since $4^3 = 64 = 8^2$, he wrote 8 as $\boxed{1^p\,\frac{1}{2}}\,4$ or $\boxed{\frac{p.1}{1.2}}\,4$, meaning $4^{1\frac{1}{2}}$. He also wrote a tract called "De latitudinibus formarum" (c. 1360), in which he graphs a dependent variable (latitudo) against an independent one (longitudo), which is subjected to variation. It shows a kind of vague transition from coordinates on the terrestrial or celestial sphere, known to the Ancients, to modern coordinate geometry. This tract was printed several times between 1482 and 1515 and may have influenced Renaissance mathematicians, including Descartes.

6. The main line of mathematical advance passed through the growing mercantile cities under the direct

[1]"A continuum cannot consist of indivisibles."

influence of trade, navigation, astronomy, and surveying. The townspeople were interested in counting, in arithmetic, in computation. Sombart had labeled this interest of the Fifteenth and Sixteenth century burgher his "Rechenhaftigkeit"[1]. Leaders in the love for practical mathematics were the reckon masters, only very occasionally joined by a university man, able, through his study of astronomy, to understand the importance of improving computational methods. Centers of the new life were the Italian cities and the Central European cities of Nuremberg, Vienna, and Prague. The fall of Constantinople in 1453, which ended the Byzantine Empire, led many Greek scholars to the Western cities. Interest in the original Greek texts increased, and it became easier to satisfy this interest. University professors joined with cultured laymen in studying the texts, ambitious reckon masters listened and tried to understand the new knowledge in their own way.

Typical of this period was Johannes Müller of Königsberg, or Regiomontanus, the leading mathematical figure of the Fifteenth Century. The activity of this remarkable computer, instrument maker, printer, and scientist illustrates the advances made in European mathematics during the two centuries after Leonardo. He was active in translating and publishing the classical mathematical manuscripts available. His teacher, the Viennese astronomer, George Peurbach—author of astronomical and trigonometrical tables—had already begun a translation of the astronomy of Ptolemy

[1]W. Sombart, *Der Bourgeois* (Munich, Leipzig, 1913), p. 164. The term "Rechenhaftigkeit" indicates a willingness to compute, a belief in the usefulness of arithmetical work. (Also London, 1915.)

from the Greek. Regiomontanus continued this translation and also translated Apollonios, Heron, and the most difficult of all, Archimedes. His main original work was "De triangulis omnimodus" (1464, not printed until 1533), a complete introduction into trigonometry, differing from our present-day texts primarily in the fact that our convenient notation did not exist. It contains the law of sines in a spherical triangle. All theorems had still to be expressed in words. Trigonometry, from now on, became a science independent of astronomy. Nasīr al-dīn had accomplished something similar in the Thirteenth Century, but it is significant that his work never resulted in much further progress, whereas Regiomontanus' book deeply influenced further development of trigonometry and its application to astronomy and algebra. Regiomontanus also devoted much effort to the computation of trigonometric tables. He has a table of sines to radius 60.000 for intervals of one minute (publ. 1490).

7. So far no definite step had been taken beyond the ancient achievements of the Greeks and Arabs. The classics remained the *nec plus ultra* of science. It came therefore as an enormous and exhilarating surprise when Italian mathematicians of the early Sixteenth Century actually showed that it was possible to develop a new mathematical theory which the Ancients and Arabs had missed. This theory, which led to the general algebraic solution of the cubic equation, was discovered by Scipio Del Ferro and his pupils at the University of Bologna.

The Italian cities had continued to show proficiency

in mathematics after the time of Leonardo. In the Fifteenth Century their reckon masters were well versed in arithmetical operations, including surds (without having any geometrical scruples) and their painters were good geometers. Vasari in his "Lives of the Painters" stresses the considerable interest which many quattrocento artists showed in solid geometry. One of their achievements was the development of perspective by such men as Alberti and Pier Della Francesca; the latter also wrote a volume on regular solids. The reckon masters found their interpreter in the Franciscan monk Luca Pacioli, whose "Summa de Arithmetica" was printed in 1494—one of the first mathematical books to be printed[1]. Written in Italian—and not a very pleasant Italian—it contained all that was known in that day of arithmetic, algebra, and trigonometry. By now the use of Hindu-Arabic numerals was well established, and the arithmetical notation did not greatly differ from ours. Pacioli ended his book with the remark that the solution of the equations $x^3 + mx = n$, $x^3 + n = mx$ was as impossible at the present state of science as the quadrature of the circle.

At this point began the work of the mathematicians at the University of Bologna. This university, around the turn of the Fifteenth Century, was one of the largest and most famous in Europe. Its faculty of astronomy alone at one time had sixteen lectors. From all parts of Europe students flocked to listen to the lectures—and to the public disputations which also attracted the

[1]The first printed mathematical books were a commercial arithmetic (Treviso, 1478) and a Latin edition of Euclid's *Elements* (Venice, Ratdolt, 1482).

LUCA PACIOLI (1450–1520) WITH THE YOUNG DUKE OF URBINO AT THE LEFT

attention of large, sportively-minded crowds. Among the students at one time or another were Pacioli, Albrecht Dürer, and Copernicus. Characteristic of the new age was the desire not only to absorb classical information but also to create new things, to penetrate beyond the boundaries set by the classics. The art of printing and the discovery of America were examples of such possibilities. Was it possible to create new mathematics? Greeks and Orientals had tried their ingenuity on the solution of the third degree equation but had only solved some special cases numerically. The Bolognese mathematicians now tried to find the general solution.

These cubic equations could all be reduced to three types:

$$x^3 + px = q, x^3 = px + q, x^3 + q = px,$$

where p and q were positive numbers. They were specially investigated by Professor Scipio Del Ferro, who died in 1526. It may be taken on the authority of E. Bortolotti[1], that Del Ferro actually solved all types. He never published his solutions and only told a few friends about them. Nevertheless, word of the discovery became known and after Scipio's death a Venetian reckon master, nicknamed Tartaglia ("The Stammerer"), rediscovered his methods (1535). He showed his results in a public demonstration, but again kept the method by which he had obtained them a secret. Finally he revealed his ideas to a learned Milanese doctor, Hieronimo Cardano, who had to swear that he

[1]E. Bortolotti, *L'algebra nella scuola matematica bolognese del secolo XVI*, Periodico di Matematica (4) 5 (1925) pp. 147-184.

would keep them a secret. But when Cardano in 1545 published his stately book on algebra, the "Ars magna," Tartaglia discovered to his disgust that the method was fully disclosed in the book, with due acknowledgement to the discoverer, but stolen just the same. A bitter debate ensued, with insults hurled both ways, in which Cardano was defended by a younger gentleman scholar, Ludovico Ferrari. Out of this war came some interesting documents, among them the "Quaesiti" of Tartaglia (1546) and the "Cartelli" of Ferrari (1547-48), from which the whole history of this spectacular discovery became public knowledge.

The solution is now known as the Cardano solution, which for the case $x^3 + px = q$ takes the form

$$x = \sqrt[3]{\sqrt{\frac{p^3}{27} + \frac{q^2}{4}} + \frac{q}{2}} - \sqrt[3]{\sqrt{\frac{p^3}{27} + \frac{q^2}{4}} - \frac{q}{2}}$$

We see that this solution introduced quantities of the form $\sqrt[3]{a + \sqrt{b}}$, different from the Euclidean $\sqrt{a + \sqrt{b}}$.

Cardano's "Ars Magna" contained another brilliant discovery: Ferrari's method of reducing the solution of the general biquadratic equation to that of a cubic equation. Ferrari's equation was $x^4 + 6x^2 + 36 = 60x$, which he reduced to $y^3 + 15y^2 + 36y = 450$. Cardano also considered negative numbers, calling them "fictitious," but was unable to do anything with the so-called "irreducible case" of the cubic equation in which there are three real solutions appearing as the sum or difference of what we now call imaginary numbers.

This difficulty was solved by the last of the great

Sixteenth Century Bolognese mathematicians, Raffael Bombelli, whose "Algebra" appeared in 1572. In this book—and in a geometry written around 1550 which remained in manuscript—he introduced a consistent theory of imaginary complex numbers. He wrote $3i$ as $\sqrt{0-9}$ (lit : $R[0\ m.\ 9]$, R for radix, m for meno). This allowed Bombelli to solve the irreducible case by showing, for instance, that

$$\sqrt[3]{52+\sqrt{0-2209}} = 4+\sqrt{0-1}.$$

Bombelli's book was widely read; Leibniz selected it for the study of cubic equations, and Euler quotes Bombelli in his own "Algebra" in the chapter on biquadratic equations. Complex numbers, from now on, lost some of their supernatural character, though full acceptance came only in the Nineteenth Century.

It is a curious fact that the first introduction of the imaginaries occurred in the theory of cubic equations, in the case where it was clear that real solutions exist though in an unrecognizable form, and not in the theory of quadratic equations, where our present textbooks introduce them.

8. Algebra and computational arithmetic remained for many decades the favorite subject of mathematical experimentation. Stimulation no longer came only from the "Rechenhaftigkeit" of the mercantile bourgeoisie but also from the demands made on surveying and navigation by the leaders of the new national states. Engineers were needed for the erection of public works and for military constructions. Astronomy remained, as in all previous periods, an important domain for

mathematical studies. It was the period of the great astronomical theories of Copernicus, Tycho Brahe, and Kepler. A new conception of the universe emerged.

Philosophical thought reflected the trends in scientific thinking; Plato with his admiration for quantitative mathematical reasoning gained ascendancy over Aristotle. Platonic influence is particularly evident in Kepler's work. Trigonometrical and astronomical tables appeared with increasing accuracy, especially in Germany. The tables of G. J. Rheticus, finished in 1596 by his pupil Valentin Otho, contain the values of all six trigonometric values for every ten seconds to ten places. The tables of Pitiscus (1613) went up to fifteen places. The technique of solving equations and the understanding of the nature of their roots also improved. The public challenge, made in 1593 by the Belgian mathematician Adriaen Van Roomen, to solve the equation of the 45th degree

$$x^{45} - 45x^{43} + 945x^{41} - 12300x^{39} + \cdots - 3795x^{3} + 45x = A,$$

was characteristic of the times. Van Roomen proposed special cases, e.g. $A = \sqrt{2+\sqrt{2+\sqrt{2+2}}}$, which gives $x = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}}$, which cases were suggested by consideration of regular polygons.

François Viète, a French lawyer attached to the court of Henry IV, solved Van Roomen's problem by observing that the left hand member was equivalent to the expression of $\sin \phi$ in terms of $\sin \phi/45$. The solution could therefore be found with the aid of tables. Viète

found twenty-three solutions of the form sin ($\phi/45 - n.8°$), discarding negative roots. Viète also reduced Cardano's solution of the cubic equation to a trigonometric one, in which process the irreducible case lost its horrors by avoiding the introduction of $\sqrt{0 - a}$. This solution can now be found in the textbooks of higher algebra.

Viète's main achievements were in the improvement of the theory of equations (e.g. "In artem analyticam isagoge", 1591), where he was among the first to represent numbers by letters. The use of numerical coefficients, even in the "rhetoric" algebra of the Diophantine school, had impeded the general discussion of algebraic problems. The work of the Sixteenth Century algebrists (the "Cossists," after the Italian word "cosa" for the unknown) was produced in a rather complicated notation. But in Viète's "logistica speciosa" at least a general symbolism appeared, in which letters were used to express numerical coefficients, the signs $+$ and $-$ were used in our present meaning and "A quadratum" was written for A^2. This algebra still differed from ours in Viète's insistence on the Greek principle of homogeneity, in which a product of two line segments was necessarily conceived as an area; line segments could therefore only be added to line segments, areas to areas, and volumes to volumes. There was even some doubt whether equations of degree higher than three actually had a meaning, since they could only be interpreted in four dimensions, a conception hard to understand in those days.

This was the period in which computational technique reached new heights. Viète improved on Archi-

Courtesy of Scripta Mathematica

FRANCOIS VIÈTE (1540–1603)

medes and found π in nine decimals; shortly afterwards π was computed in thirty-five decimals by Ludolph Van Coolen, a fencing master at Delft who used inscribed and circumscribed regular polygons with more and more sides. Viète also expressed π as an infinite product (1593, in our notation):

$$\frac{2}{\pi} = \cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cos\frac{\pi}{32}\cdots.$$

The improvement in technique was a result of the improvement in notation. The new results show clearly that it is incorrect to say that men like Viète "merely" improved notation. Such a statement discards the profound relation between content and form. New results have often become possible only because of a new mode of writing. The introduction of Hindu-Arabic numerals is one example; Leibniz' notation for the calculus is another one. An adequate notation reflects reality better than a poor one, and as such appears endowed with a life of its own which in turn creates new life. Viète's improvement in notation was followed, a generation later, by Descartes' application of algebra to geometry.

9. Engineers and arithmeticians were in particular demand in the new commercial states, especially France, England, and the Netherlands. Astronomy flourished over all Europe. Although the Italian cities were no longer on the main road to the Orient after the discovery of the sea route to India, they still remained centers of importance. And so we find among the great mathematicians and computers of the early Seventeenth Century Simon Stevin, an engineer, Johann Kepler, an astronomer, and Adriaan Vlacq and Ezechiel De Decker, surveyors.

Stevin, a bookkeeper of Bruges, became an engineer in the army of Prince Maurice of Orange, who appreciated the way Stevin combined practical sense with theoretical understanding and originality. In "La disme" (1585) he introduced decimal fractions as part of a project to unify the whole system of measurements on a decimal base. It was one of the great improvements made possible by the general introduction of the Hindu-Arabic system of numeration.

The other great computational improvement was the invention of logarithms. Several mathematicians of the Sixteenth Century had been playing with the possibility of coordinating arithmetical and geometrical progressions, mainly in order to ease the work with the complicated trigonometrical tables. An important contribution toward this end was undertaken by a Scottish laird, John Neper (or Napier), who in 1614 published his "Mirifici logarithmorum canonis descriptio." His central idea was to construct two sequences of numbers so related that when one increases in arithmetical progression, the other decreases in a geometrical one. Then the product of two numbers in the second sequence has a simple relation to the sum of corresponding numbers in the first, and multiplication could be reduced to addition. With this system Neper could considerably facilitate computational work with sines. Neper's early attempt was rather clumsy, since his two sequences correspond according to the modern formula

$$y = a\,e^{x/a} \text{ (or } x = \text{Nep. log } y)$$

JOHN NAPIER (1550–1617)

in which $a = 10^7$.[1] When $x = x_1 + x_2$, we do not get $y = y_1y_2$, but $y = y_1y_2/a$. This system did not satisfy Neper himself, as he told his admirer Henry Briggs, a professor at Gresham College, London. They decided on the function $y = 10^x$, for which $x = x_1 + x_2$ actually yields $y = y_1y_2$. Briggs, after Neper's death, carried out this suggestion and in 1624 published his "Arithmetica logarithmica" which contained the "Briggian" logarithms in 14 places for the integers from 1 to 20.000 and from 90.000 to 100.000. The gap between 20.000 and 90.000 was filled by Ezechiel De Decker, a Dutch surveyor, who assisted by Vlacq, published at Gouda in 1627 a complete table of logarithms. The new invention was immediately welcomed by the mathematicians and astronomers, and particularly by Kepler, who had had a long and painful experience with elaborate computations.

Our explanation of logarithms by exponentials is historically somewhat misleading, since the conception of an exponential function dates only from the later part of the Seventeenth Century. Neper had no notion of a base. Natural logarithms, based on the function $y = e^x$, appeared almost contemporaneously with the Briggian logarithms, but their fundamental importance was not recognized until the infinitesimal calculus was better understood[2].

[1]Hence Nep. log $y = 10^7 (\ln 10^7 - \ln y) = 161180957 - 10^7 \ln y$; and Nep. log $1 = 161180957$; $\ln x$ stands for our natural logarithm.

[2]E. Wright published some natural logarithms in 1618, J. Speidel in 1619, but after this no tables of these logarithms were published until 1770. See F. Cajori, *History of the Exponential and Logarithmic Concepts*, American Math. Monthly 20 (1913).

Literature.

For the spread of Hindu-Arabic numerals in Europe see:

D. E. Smith—L. C. Karpinski, *The Hindu-Arabic Numerals* (Boston, London, 1911)

For speculative mathematics in the Middle Ages see:

C. B. Boyer, *The Concepts of the Calculus*, Chapter III (London, 1939), 346 pp.

Sixteenth and Seventeenth Century Italian mathematics is discussed in a series of papers by:

E. Bortolotti, written between 1922-28, e.g. Periodico di matematica 5 (1925) pp. 147-184; 6 (1926) pp. 217-230; 8 (1928) pp. 19-59; Scientia 1923, pp. 385-394 and

E. Bortolotti, *I contributi del Tartaglia, del Cardano, del Ferrari, e della scuola matematica bolognese alla teoria algebrica delle equazioni cubiche*, Imola (1926) 54 pp.

Cardano's autobiography has been translated:

G. Cardano, *The Book of My Life*, transl. by J. Stoner (London, 1931).

Much information on Sixteenth and Seventeenth Century mathematicians and their works is in the papers of H. Bosmans S.J., most of which can be found in the Annales de la Société Scientifique Bruxelles,

1905-27. Complete list in A. Rome, Isis 12 (1929) p. 88. Furthermore:

H. S. Carslaw, *The Discovery of Logarithms by Napier*, Mathem. Gazette. (1915-1916) pp. 76-84, 115-119.

E. Zinner, *Leben und Wirken des Johannes Müller von Königsberg genannt Regiomontanus* (Munich, 1938).

J. D. Bond, *The Development of Trigonometric Methods down to the close of the Fifteenth Century*, Isis 4 (1921-22) pp. 295-323.

F. A. Yeldham, *The Story of Reckoning in the Middle Ages* (London, 1926).

E. J. Dÿksterhuis, *Simon Stevin* ('s Gravenhage, 1943).

CHAPTER VI

The Seventeenth Century

1. The rapid development of mathematics during the Renaissance was due not only to the "Rechenhaftigkeit" of the commercial classes but also to the productive use and further perfection of machines. Machines were known to the Orient and to classical antiquity; they had inspired the genius of Archimedes. However, the existence of slavery and the absence of economically progressive urban life frustrated the use of machines in these older forms of society. This is indicated by the works of Heron, where machines are described, but only for the purpose of amusement or deception.

In the later Middle Ages machines came in use in small manufactures, in public works, and in mining. These were enterprises undertaken by city merchants or by princes in search of ready money and often conducted in opposition to the city guilds. Warfare and navigation also stimulated the perfection of tools and their further replacement by machines.

A well-established silk industry existed in Lucca and in Venice as early as the Fourteenth Century. It was based on division of labor and on the use of water power. In the Fifteenth Century mining in Central Europe developed into a completely capitalistic industry based technically on the use of pumps and hoisting machines which allowed the boring of deeper and deeper layers. The invention of firearms, and of printing, the construction of windmills and canals, the building of ships

to sail the ocean, required engineering skill and made people technically conscious. The perfection of clocks, useful for astronomy and navigation and often installed in public places, brought admirable pieces of mechanism before the public eye; the regularity of their motion and the possibility they offered of indicating time exactly made a deep impression upon the philosophical mind. During the Renaissance and even centuries later the clock was taken as a model of the universe. This was an important factor in the development of the mechanical conception of the world.

Machines led to theoretical mechanics and to the scientific study of motion and of change in general. Antiquity had already produced texts on statics, and the new study of theoretical mechanics naturally based itself upon the statics of the classical authors. Books on machines appeared long before the invention of printing, first empirical descriptions (Kyeser, early Fifteenth Century), later more theoretical ones, such as Leon Battista Alberti's book on architecture (c. 1450) and the writings of Leonardo Da Vinci (c. 1500). Leonardo's manuscripts contain the beginnings of a definite mechanistic theory of nature. Tartaglia in his "Nuova scienzia" (1537) discussed the construction of clocks and the orbit of projectiles—but had not yet found the parabolic orbit, first discovered by Galileo. The publication of Latin editions of Heron and Archimedes stimulated this kind of research, especially F. Commandino's edition of Archimedes, which appeared in 1558 and brought the ancient method of integration within the reach of the mathematicians. Commandino himself applied these methods to the computation of

centers of gravity (1565), though with less rigor than his master.

This computation of centers of gravity remained a favorite topic of Archimedian scholars, who used their study of statics to obtain a working knowledge of the rudiments of what we now recognize as the calculus. Outstanding among such students of Archimedes is Simon Stevin who wrote on centers of gravity and on hydraulics both in 1586, Luca Valerio who wrote on centers of gravity in 1604 and on the quadrature of the parabola in 1606, and Paul Guldin in whose "Centrobaryca" (1641) we find the so-called theorem of Guldin on centroids, already explained by Pappus. In the wake of the early pioneers came the great works of Kepler, Cavalieri, and Torricelli, in which they evolved methods which eventually led to the invention of the calculus.

2. Typical of these authors was their willingness to abandon Archimedian rigor for considerations often based on non-rigorous, sometimes "atomic," assumptions—probably without knowing that Archimedes, in his letter to Eratosthenes, had also used such methods for their heuristic value. This was partly due to impatience with scholasticism among some, though not all, of these authors, since several of the pioneers were Catholic priests trained in scholasticism. The main reason was the desire for results, which the Greek method was unable to provide quickly.

The revolution in astronomy, connected with the names of Copernicus, Tycho Brahe, and Kepler, opened entirely new visions of man's place in the universe and man's power to explain the phenomena of astronomy

Courtesy of Scripta Mathematica

JOHANN KEPLER (1571–1630)

in a rationalistic way. The possibility of a celestial mechanics to supplement terrestrial mechanics increased the boldness of the men of science. In the works of Johann Kepler the stimulating influence of the new astronomy on problems involving large computations as well as infinitesimal considerations is particularly evident. Kepler even ventured into volume computation for its own sake, and in his "Stereometria doliorum vinorum" ("Solid geometry of wine barrels", 1615) evaluated the volumes of solids obtained by rotating segments of conic sections about an axis in their plane. He broke with Archimedian rigor; his circle area was composed of an infinity of triangles with common vertex at the center; his sphere consisted of an infinity of pointed pyramids. The proofs of Archimedes, Kepler said, were absolutely rigorous, "absolutae et omnibus numeris perfectae"[1], but he left them to the people who wished to indulge in exact demonstrations. Each successive author was free to find his own kind of rigor, or lack of rigor, for himself.

To Galileo Galilei we owe the new mechanics of freely falling bodies, the beginning of the theory of elasticity, and a spirited defense of the Copernican system. Above all we owe to Galileo, more than to any other man of his period, the spirit of modern science based on the harmony of experiment and theory. In the "Discorsi" (1638) Galileo was led to the mathematical study of motion, to the relation between distance, velocity, and acceleration. He never gave a systematic explanation of his ideas on the calculus, leaving this to his pupils Torricelli and Cavalieri. Indeed,

[1] "absolute and in all respects perfect"

Courtesy of Scripta Mathematica

GALILEO (1564–1642)

Galileo's ideas on these questions of pure mathematics were quite original, as appears from his remark that "neither is the number of squares less than the totality of all numbers, nor the latter greater than the former." This defense of the actually infinite (given by Salviati in the "Discorsi") was consciously directed against the Aristotelian and scholastic position (represented by Simplicio). The "Discorsi" also contain the parabolic orbit of the projectile, with tables for height and range as functions of the angle of elevation and given initial velocity. Salviati also remarks that the catenary looks like a parabola, but does not give the precise description of the curve.

The time had now arrived for a first systematic exposition of the results reached so far in what we now call the calculus. This exposition appeared in the "Geometria indivisibilibus continuorum" (1635) of Bonaventura Cavalieri, professor at the University of Bologna. Here Cavalieri established a simple form of the calculus, basing it on the scholastic conception of the "indivisibile"[1], the point generating the line, the line generating the plane by motion. Cavalieri, therefore, had no infinitesimals or "atoms." He came to his results by the "principle of Cavalieri," which concludes that two solids of equal altitudes have the same volume, if plane cross sections at equal height have the same area. It

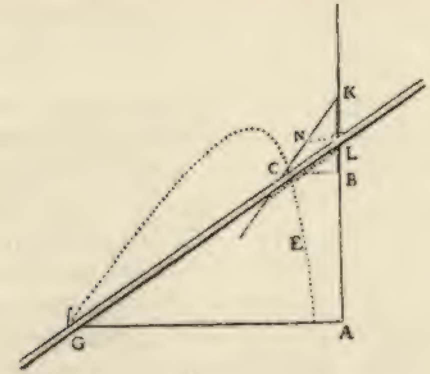
[1]F. Cajori, *Indivisibles and "ghosts of departed quantities" in the History of Mathematics*, Scientia 1925, pp. 301-306; E. Hoppe, *Zur Geschichte der Infinitesimalrechnung bis Leibniz und Newton*, Jahresb. Deutsch. Math. Verein.37 (1928) pp. 148-187. On certain statements in Hoppe see C. B. Boyer, l.c., pp. 192, 206, 209.

allowed him to perform the equivalent of the integration of polynomials.

3. This gradual evolution of the calculus was considerably stimulated by the publication of Descartes' "Géométric" (1637), which brought the whole field of classical geometry within the scope of the algebrists. The book was originally published as an appendix to the "Discours de la Méthode", the discourse on reason, in which the author explained his rationalistic approach to the study of nature. Réné Descartes was a Frenchman from Touraine, lived the life of a gentleman, served for a while in the army of Maurice of Orange, stayed for many years in the Netherlands, and died in Stockholm to which city he had been invited by the Queen of Sweden. In accordance with many other great thinkers of the Seventeenth Century, Descartes searched for a general method of thinking in order to be able to facilitate inventions and to find the truth in the sciences. Since the only known natural science with some degree of systematic coherence was mechanics, and the key to the understanding of mechanics was mathematics, mathematics became the most important means for the understanding of the universe. Moreover, mathematics with its convincing statements was itself the brilliant example that truth could be found in science. The mechanistic philosophy of this period thus came to a conclusion which was similar to that of the Platonists, but for a different reason. Platonists, believing in authority, and Cartesians, believing in reason, both found in mathematics the queen of the sciences.

Descartes published his "Géométrie" as an applica-

LIVRE SECOND.



Aprés cela prenant vn point a diſcretion dans la courbe, comme C, ſur lequel ie ſuppoſe que l'inſtrument qui ſert a la deſcrire eſt appliqué, ie tire de ce point C la ligne C B parallele a G A, & pourceque C B & B A ſont deux quantités indeterminées & inconnuës, ie les nomme l'vne y & l'autre x. mais affin de trouuer le rapport de l'vne à l'autre; ie conſidere auſſy les quantités connuës qui determinent la deſcription de cete ligne courbe, comme G A que ie nomme a, K L que ie nomme b, & N L parallele a G A que ie nomme c. puis ie dis, comme N L eſt à L K, ou c à b, ainſi C B, ou y, eſt à B K, qui eſt par conſequent $\frac{b}{c}y$: & B L eſt $\frac{b}{c}y - b$, & A L eſt $x + \frac{b}{c}y - b$. de plus comme C B eſt à L B, ou y à $\frac{b}{c}y - b$, ainſi a, ou G A, eſt à L A, ou $x + \frac{b}{c}y - b$. de façon que mul-

S f tipliant

A FACSIMILE PAGE OF DESCARTES' *La Géométrie*

tion of his general method of unification, in this case the unification of algebra and geometry. The merits of the book, according to the commonly accepted point of view, consist mainly in the creation of so-called analytical geometry. It is true that this branch of mathematics eventually evolved under the influence of Descartes' book, but the "Géométrie" itself can hardly be considered a first textbook on this subject. There are no "Cartesian" axes and no equations of the straight line and of conic sections are derived, though a particular equation of the second degree is interpreted as denoting a conic section. Moreover, a large part of the book consists of a theory of algebraic equations, containing the "rule of Descartes" to determine the number of positive and negative roots.

We must keep in mind that Apollonios already had a characterization of conic sections by means of what we now—with Leibniz—call coordinates, and Pappos had in his "Collection" a "Treasury of Analysis" ("Analyomenos"), in which we have only to modernize the notation to obtain a consistent application of algebra to geometry. Even graphical representations occur before Descartes (Oresme). Descartes' merits lie above all in his consistent application of the well developed algebra of the early Seventeenth Century to the geometrical analysis of the Ancients, and by this, in an enormous widening of its applicability. A second merit is Descartes' final rejection of the homogeneity restrictions of his predecessors which even vitiated Viète's "logistica speciosa," so that x^2, x^3, xy were now considered as line segments. An algebraic equation became a relation between numbers, a new advance in

RENÉ DESCARTES (1596–1650)

mathematical abstraction necessary for the general treatment of algebraic curves.

Much in Descartes' notation is already modern; we find in his book expressions such as $\frac{1}{2}a + \sqrt{\frac{1}{4}aa + bb}$, which differs from our own notation only in Descartes' still writing aa for a^2 (which is even found in Gauss), though he has a^3 for aaa, a^4 for $aaaa$, etc. It is not hard to find one's way in his book, but we must not look for our modern analytical geometry.

A little closer to such analytical geometry came Pierre Fermat, a lawyer at Toulouse, who wrote a short paper on geometry probably before the publication of Descartes' book, but which was only published in 1679. In this "Isagoge" we find the equations $y = mx$, $xy = k^2$, $x^2 + y^2 = a^2$, $x^2 \pm a^2y^2 = b^2$ assigned to lines and conics, with respect to a system of (usually perpendicular) axes. However, since it was written in Viète's notation, the paper looks more archaic than Descartes' "Géométrie." At the time when Fermat's "Isagoge" was printed there were already other publications in which algebra was applied to Apollonios' results, notably the "Tractatus de sectionibus conicis" (1655) by John Wallis and a part of the "Elementa curvarum linearum" (1659) written by Johan De Witt, grand pensionary of Holland. Both these works were written under the direct influence of Descartes. But progress was very slow; even L'Hospital's "Traité analytique des sections coniques" (1707) has not much more than a transcription of Apollonios into algebraic language. All authors hesitated to accept negative values for the coordinates. The first to work boldly with algebraic

PASCAL EXPLAINS TO DESCARTES HIS PLANS FOR EXPERIMENTS IN WEIGHT
(From a painting by Chartran in the Sorbonne)
The group in the right hand corner consists of Desargues, Mersenne, Pascal and Descartes.

equations was Newton in his study of cubic curves (1703); the first analytic geometry of conic sections which is fully emancipated from Apollonios appeared only with Euler's "Introductio" (1748).

4. The appearance of Cavalieri's book stimulated a considerable number of mathematicians in different countries to study problems involving infinitesimals. The fundamental problems began to be approached in a more abstract form and in this way gained in generality. The tangent problem, consisting in the search for methods to find the tangent to a given curve at a given point, took a more and more prominent place beside the ancient problems involving volumes and centers of gravity. In this search there were two marked trends, a geometrical and an algebraic one. The followers of Cavalieri, notably Torricelli and Isaac Barrow, Newton's teacher, followed the Greek method of geometrical reasoning without caring too much about its rigor. Christiaan Huygens also showed a definite partiality for Greek geometry. There were others, notably Fermat, Descartes, and John Wallis, who showed the opposite trend and brought the new algebra to bear upon the subject. Practically all authors in this period from 1630 to 1660 confined themselves to questions dealing with algebraic curves, especially those with equation $a^m y^n = b^n x^m$. And they found, each in his own way, formulas equivalent to $\int_0^a x^m\,dx = a^{m+1}/(m+1)$, first for positive integer m, later for m negative integer and fractional. Occasionally a non-algebraic curve appeared, such as the cycloid (roulette) investigated by Descartes and Blaise Pascal; Pascal's

"Traité général de la roulette" (1658), a part of a booklet published under the name of A. Dettonville, had great influence on young Leibniz.[1]

In this period several characteristic features of the calculus began to appear. Fermat discovered in 1638 a method to find maxima and minima by changing slightly the variable in a simple algebraic equation and then letting the change disappear; it was generalized in 1658 to more general algebraic curves by Johannes Hudde, a burgomaster of Amsterdam. There were determinations of tangents, volumes, and centroids, but the relation between integration and differentiation as inverse problems was not really grasped (except perhaps by Barrow). Pascal occasionally used expansions in term of small quantities in which he dropped the terms of lower dimensions—anticipating the debatable assumption of Newton that $(x + dx)(y + dy) - xy = xdy + ydx$. Pascal defended his procedure by appealing to intuition ("esprit de finesse") rather than to logic ("esprit de géométrie"), here anticipating Bishop Berkeley's criticism of Newton.[2]

Scholastic thought entered into this search for new methods not only through Cavalieri but also through the work of the Belgian Jesuit, Grégoire De Saint Vincent and his pupils and associates, Paul Guldin and André Tacquet. These men were inspired by both the spirit of their age and the medieval scholastic writings on the nature of the continuum and the latitude of

[1] H. Bosmans, *Sur l'oeuvre mathématique de Blaise Pascal*, Revue des Questions Scientifiques (1929), 63 pp.

[2] B. Pascal, *Oeuvres* (Paris, 1908-1914) XII., p. 9, XIII, pp. 141-155.

forms. In their writings the term "exhaustion" for Archimedes' method appears for the first time. Tacquet's book, "On Cylinders and Rings" (1651) influenced Pascal.

This fervid activity of mathematicians in a period when no scientific periodicals existed led to discussion circles and to constant correspondence. Some figures gained merit by serving as a center of scientific interchange. The best known of these men is the Minorite Father Marin Mersenne, whose name as a mathematician is preserved in Mersenne's numbers. With him corresponded Descartes, Fermat, Desargues, Pascal, and many other scientists.[1] Academies crystallized out of the discussion groups of learned men. They arose in a way as opposition to the universities which had developed in the scholastic period—with some exceptions such as Leiden University—and still fostered the medieval attitude of presenting knowledge in fixed forms. The new academies, on the contrary, expressed the new spirit of investigation. They typified "this age drunk with the fulness of new knowledge, busy with the uprooting of superannuated superstitions, breaking loose from traditions of the past, embracing most extravagant hopes for the future. Here the individual scientist learned to be contented and proud to have added an infinitesimal part to the sum of knowledge; here, in short, the modern scientist was evolved."[2]

[1] "Informer Mersenne d'une découverte, c'était la publier par l'Europe entière," writes H. Bosmans (l.c., p. 43: "To inform Mersenne of a discovery, meant to publish it throughout the whole of Europe").

[2] M. Ornstein, *The Role of Scientific Societies in the Seventeenth Century* (Chicago, 1913).

The first academy was founded in Naples (1560); it was followed by the "Accademia dei Lincei" in Rome (1603). The Royal Society dates from 1662, the French Academy from 1666. Wallis was a charter member of the Royal Society, Huygens of the French Academy.

5. Next to Cavalieri's one of the most important books written in this period of anticipation is Wallis' "Arithmetica infinitorum" (1655). The author was from 1643 until his death in 1703 the Savilian professor of geometry at Oxford. Already the title of his book shows that Wallis intended to go beyond Cavalieri with his "Geometria indivisibilium"; it was the new "arithmetica" (algebra) which Wallis wanted to apply, not the ancient geometry. In this process Wallis extended algebra into a veritable analysis—the first mathematician to do so. His methods of dealing with infinite processes were often crude, but he obtained new results; he introduced infinite series and infinite products and used with great boldness imaginaries, negative and fractional exponents. He wrote ∞ for $\frac{1}{0}$ (and claimed that $-1 > \infty$). Typical of his results is the expansion

$$\frac{\pi}{2} = \frac{2\cdot 2\cdot 4\cdot 4\cdot 6\cdot 6\cdot 8\cdot 8\ \cdots}{1\cdot 3\cdot 3\cdot 5\cdot 5\cdot 7\cdot 7\cdot 9\ \cdots}$$

Wallis was only one of a whole series of brilliant men of this period, who enriched mathematics with discovery after discovery. The driving force behind this flowering of creative science, unequalled since the great days of Greece, was only in part the ease with which the new techniques could be handled. Many great thinkers were

in search of more: of a "general method"—sometimes conceived in a restricted sense as a method of mathematics, sometimes more general as a method of understanding nature and of creating new inventions. This is the reason why in this period all outstanding philosophers were mathematicians and all outstanding mathematicians philosophers. The search for new inventions sometimes led directly to mathematical discoveries. A famous example is the "Horologium oscillatorium" (1673) of Christiaan Huygens, where the search for better timepieces led not only to pendulum clocks but also to the study of evolutes and involutes of a plane curve. Huygens was a Hollander of independent means who stayed for many years in Paris; he was eminent as a physicist as well as an astronomer, established the wave theory of light, and explained that Saturn had a ring. His book on pendulum clocks was of influence on Newton's theory of gravitation; it represents with Wallis' "Arithmetica" the most advanced form of the calculus in the period before Newton and Leibniz. The letters and books of Wallis and of Huygens abound in new discoveries, in rectifications, envelopes, and quadratures. Huygens studied the tractrix, the logarithmic curve, the catenary, and established the cycloid as a tautochronous curve. Despite this wealth of results, many of which were found after Leibniz had published his calculus, Huygens belongs definitely to the period of anticipation. He confessed to Leibniz that he never was able to familiarize himself with Leibniz' method. Wallis, in the same way, never found himself at home in Newton's notation. Huygens was one of the few great Seventeenth Century mathematicians who took rigor

CHRISTIAAN HUYGENS (1629–1695)

seriously; his methods were always strictly Archimedian.

6. The activity of the mathematicians of this period stretched into many fields, new and old. They enriched classical topics with original results, cast new light upon ancient fields, and even created entirely new subjects of mathematical research. An example of the first case was Fermat's study of Diophantos; an example of the second case was Desargues' new interpretation of geometry. The mathematical theory of probability was an entirely new creation.

Diophantos became available to a Latin reading public in 1621[1]. In Fermat's copy of this translation are found his famous marginal notes, which his son published in 1670. Among them we find Fermat's "great theorem" that $x^n + y^n = z^n$ is impossible for positive integer values of x, y, z, n if $n > 2$, which led Kummer in 1847 to his theory of ideal numbers. A proof valid for all n has not yet been given, though the theorem is certainly correct for a large number of values.[2]

Fermat wrote in the margin beside Diophantos II 8: "To divide a square number into two other square numbers," the following words: "To divide a cube into two other cubes, a fourth power, or in general any power whatever into two powers of the same denomination above the second is impossible, and I have assuredly found an admirable proof of this, but the margin is too narrow to contain it." If Fermat had such

[1]First readily available Latin translations: Euclid 1482; Ptolemy 1515; Archimedes 1558; Apollonios I-IV, 1566, V-VII 1661; Pappos 1589; Diophantos 1621.

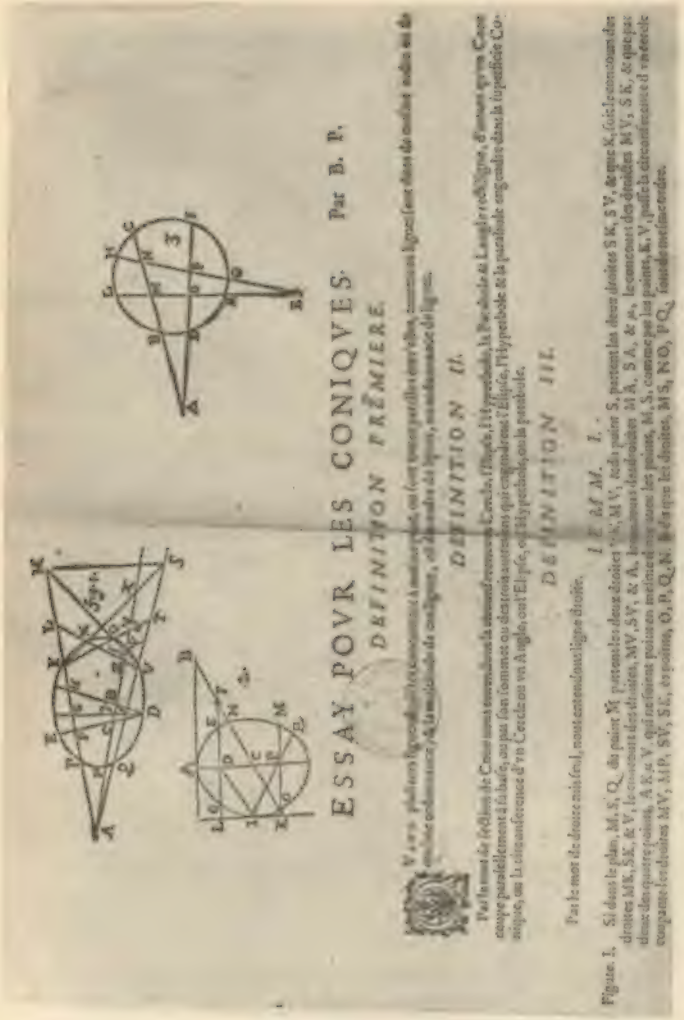
[2]See H. S. Vandiver, Am. Math. Monthly, 53 (1946) pp. 555–78.

admirable proof, then three centuries of intense research have failed to produce it again. It is safer to assume that even the great Fermat slept sometimes.

Another marginal note of Fermat states that a prime of form $4n + 1$ can be expressed once, and only once, as the sum of two squares, which theorem was later demonstrated by Euler. The other "theorem of Fermat," which states that $a^{p-1} - 1$ is divisible by p when p is prime and a is prime to p, appears in a letter of 1640; this theorem can be demonstrated by elementary means. Fermat was also the first to assert that the equation $x^2 - Ay^2 = 1$ (A a non-square integer) has an unlimited number of integer solutions.

Fermat and Pascal were the founders of the mathematical theory of probabilities. The gradual emergence of the interest in problems relating to probabilities is primarily due to the development of insurance, but the specific questions which stimulated great mathematicians to think about this matter came from requests of noblemen gambling in dice or cards. In the words of Poisson: "Un problème relatif aux jeux de hasard, proposé à un austère janséniste par un homme du monde, a été l'origine du calcul des probabilités[1]". This "man of the world" was the Chevalier De Méré, who approached Pascal with a question concerning the so-called "problème des points". Pascal began a correspondence with Fermat on this problem and on related questions, and both men established some of the founda-

[1]"A problem concerning games of chance, proposed by a man of the world to an austere Jansenist, was the origin of the calculus of probabilities" (S. D. Poisson, *Recherches sur la probabilité des jugements*, 1837, p. 1).



ESSAY POVR LES CONIQVES. Par B. P.

DEFINITION PREMIERE.

DEFINITION II.

DEFINITION III.

LEMME I.

Greatly reduced reproduction of the leaflet which BLAISE PASCAL published in 1640 as his "Theorem of Pascal"
(As shown by L. Brunschvicg, P. Boutroux, 1908)

BLAISE PASCAL (1623–1662)

tions of the theory of probability (1654). When Huygens came to Paris he heard of this correspondence and tried to find his own answers; the result was the "De ratiociniis in ludo aleae" (1657), the first treatise on probability. The next steps were taken by De Witt and Halley, who constructed tables of annuities (1671, 1693).

Blaise Pascal was the son of Etienne Pascal, a correspondent of Mersenne; the "limaçon of Pascal" is named after Etienne. Blaise developed rapidly under his father's tutelage and at the age of sixteen discovered "Pascal's theorem" concerning a hexagon inscribed in a conic. It was published in 1641 on a single sheet of paper and showed the influence of Desargues. A few years later Pascal invented a computing machine. At the age of twenty-five he decided to live the ascetic life of a Jansenist in the convent of Port Royal, but continued to devote time to science and to literature. His treaty on the "arithmetical triangle" formed by the binomial coefficients and useful in probability appeared posthumously in 1664. We have already mentioned his work on integration and his speculations on the infinitesimal, which influenced Leibniz.

Gérard Desargues was an architect from Lyons and the author of a book on perspective (1636). His pamphlet with the curious title of "Brouillon projet d'une atteinte aux évenements des rencontres d'un cone avec un plan"[1] (1639) contains in a curious botanical language some of the fundamental conceptions of synthetic geometry, such as the points at infinity, involutions, polarities. His "Desargues' theorem" on perspective

[1] "Proposed draft of an attempt to deal with the events of the meeting of a cone with a plane."

triangles was published in 1648. These ideas did not show their full fertility until the Nineteenth Century.

7. A general method of differentiation and integration, derived in the full understanding that one process is the inverse of the other, could only be discovered by men who mastered the geometrical method of the Greeks and of Cavalieri, as well as the algebraic method of Descartes and Wallis. Such men could have appeared only after 1660, and they actually did appear in Newton and Leibniz. Much has been written about the priority of the discovery, but it is now established that both men found their methods independently of each other. Newton had the calculus first (Newton in 1665-66; Leibniz in 1673-76), but Leibniz published it first (Leibniz 1684-86; Newton 1704-1736). Leibniz' school was far more brilliant than Newton's school.

Isaac Newton was the son of a country squire in Lincolnshire, England. He studied at Cambridge under Isaac Barrow who in 1669 yielded the Lucasian professorship to his pupil—a remarkable academic event since Barrow frankly acknowledged Newton to be his superior. Newton stayed at Cambridge until 1696 when he accepted the position of warden, and later of master, of the mint. His tremendous authority is primarily based on his "Philosophiae naturalis principia mathematica" (1687), an enormous tome establishing mechanics on an axiomatic foundation and containing the law of gravitation—the law which brings the apple to the earth and keeps the moon moving around the earth. He showed by rigorous mathematical deduction how the empirically established laws of Kepler on planetary

SIR ISAAC NEWTON (1642–1727)

motion were the result of the gravitational law of inverse squares and gave a dynamical explanation of many aspects of the motions of heavenly bodies and of the tides. He solved the two-body problem for spheres and laid the beginnings of a theory of the moon's motion. By solving the problem of the attraction of spheres he also laid the foundation of potential theory. His axiomatic treatment postulated absolute space and absolute time.

The geometrical form of the demonstrations hardly shows that the author was in full possession of the calculus, which he called the "theory of fluxions." Newton discovered his general method during the years 1665-66 when he stayed at his birthplace in the country to escape from the plague which infested Cambridge. From this period also date his fundamental ideas on universal gravitation, as well as the law of the composition of light. "There are no other examples of achievement in the history of science to compare with that of Newton during those two golden years," remarks Professor More[1].

Newton's discovery of "fluxions" was intimately connected with his study of infinite series in Wallis' "Arithmetica." It brought him to extend the binomial theorem to fractional and negative exponents and thus to the discovery of the binomial series. This again helped him greatly in establishing his theory of fluxions to "all" functions, whether algebraic or transcendental. A "fluxion", expressed by a dot placed over a letter ("pricked

[1]L. T. More, *Isaac Newton. A Biography* (N. Y., London, 1934) p. 41.

letters") was a finite value, a velocity; the letters without the dot represented "fluents."

Here is an example of the way in which Newton explained his method ("Method of Fluxions", 1736): The variables of fluents are denoted by $v, x, y, z, \cdots$ "and the velocities by which every fluent is increased by its generating motion (which I may call *fluxions*, or simply velocities, or celerities), I shall represent by the same letters pointed, thus $\dot{v}, \dot{x}, \dot{y}, \dot{z}$." Newton's infinitesimals are called "moments of fluxions," which are represented by $\dot{v}o, \dot{x}o, \dot{y}o, \dot{z}o$, o being "an infinitely small quantity." Newton then proceeds:

"Thus let any equation $x^3 - ax^2 + axy - y^3 = 0$ be given, and substitute $x + \dot{x}o$ for x, $y + \dot{y}o$ for y, and there will arise

$$x^3 + 3x^2\dot{x}o + 3x\dot{x}o\dot{x}o + \dot{x}^3o^3 - ax^2 - 2ax\dot{x}o$$
$$- a\dot{x}o\dot{x}o + axy + ay\dot{x}o + a\dot{x}o\dot{y}o + ax\dot{y}o$$
$$- y^3 - 3y^2\dot{y}o - 3y\dot{y}o\dot{y}o - \dot{y}^3o^3 = 0$$

"Now, by supposition, $x^3 - ax^2 + axy - y^3 = 0$, which therefore, being expunged and the remaining terms being divided by o, there will remain

$$3x^2\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} + 3x\dot{x}\dot{x}o - a\dot{x}\dot{x}o$$
$$+ a\dot{x}\dot{y}o - 3y\dot{y}\dot{y}o + \dot{x}^3oo - \dot{y}^3oo = 0.$$

"But whereas zero is supposed to be infinitely little, that it may represent the moments of quantities, the terms that are multiplied by it will be nothing in respect

to the rest; I therefore reject them, and there remains

$$3x^2\dot{x} - 2ax\dot{x} + ay\dot{x} + ax\dot{y} - 3y^2\dot{y} = 0."$$

This example shows that Newton thought of his derivatives primarily as velocities, but it also shows that there was a certain vagueness in his mode of expression. Are the symbols "o" zeros? are they infinitesimals? or are they finite numbers? Newton has tried to make his position clear by the theory of "prime and ultimate ratios," which he introduced in the "Principia" and which involved the conception of limit but in such a way that it was very hard to understand it.

"Those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits toward which the ratios of quantities, decreasing without limit, do always converge, and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, until the quantities have diminished in infinitum". (Principia I, Sect. I, last scholium).

"Quantities, and the ratio of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal" (1B, I, I, Lemma I).

This was far from clear, and the difficulties which the understanding of Newton's theory of fluxions involved led to much confusion and severe criticism by Bishop Berkeley in 1734. The misunderstandings were not removed until the modern limit concept was well established.

Newton also wrote on conics and plane cubic curves. In the "Enumeratio linearum tertii ordinis" (1704) he gave a classification of plane cubic curves into 72

species, basing himself on his theorem that every cubic can be obtained from a "divergent parabola" $y^2 = ax^3 + bx^2 + cx + d$ by central projection from one plane upon another. This was the first important new result reached by the application of algebra to geometry, all previous work being simply the translation of Apollonios into algebraic language. Another contribution of Newton was his method of finding approximations to the roots of numerical equations, which he explained on the example $x^3 - 2x - 5 = 0$, which yields $x = 2.09455147$.

The difficulty in estimating Newton's influence on his contemporaries lies in the fact that he always hesitated to publish his discoveries. He first tested the law of universal gravitation in 1665-66, but did not announce it until he presented the manuscript of most of the "Principia" to the printer (1686). His "Arithmetica universalis," consisting of lectures on algebra delivered between 1673-1683, was published in 1707. His work on series, which dates from 1669, was announced in a letter to Oldenburg in 1676 and appeared in print in 1711. His quadrature of curves, of 1671, was not published until 1704; this was the first time that the theory of fluxions was placed before the world. His "Method of Fluxions" itself only appeared after his death in 1736.

8. Gottfried Wilhelm Leibniz was born in Leipzig and spent most of his life near the court of Hanover in the service of the dukes, of which one became King of England under the name of George I. He was even more catholic in his interests than the other great thinkers of his century; his philosophy embraced history, theology,

GOTTFRIED WILHELM LEIBNIZ (1646–1716)
From a picture in the Uffizi Gallery, Florence

linguistics, biology, geology, mathematics, diplomacy, and the art of inventing. He was one of the first after Pascal to invent a computing machine; he imagined steam engines, studied Sanskrit, and tried to promote the unity of Germany. The search for a universal method by which he could obtain knowledge, make inventions, and understand the essential unity of the universe was the mainspring of his life. The "scientia generalis" he tried to build had many aspects, and several of them led Leibniz to discoveries in mathematics. His search for a "characteristica generalis" led to permutations, combinations, and symbolic logic; his search for a "lingua universalis", in which all errors of thought would appear as computational errors, led not only to symbolic logic but also to many innovations in mathematical notation. Leibniz was one of the greatest inventors of mathematical symbols. Few men have understood so well the unity of form and content. His invention of the calculus must be understood against this philosophical background; it was the result of his search for a "lingua universalis" of change and of motion in particular.

Leibniz found his new calculus between 1673 and 1676 under the personal influence of Huygens and by the study of Descartes and Pascal. He was stimulated by his knowledge that Newton was reported to be in the possession of such a method. Where Newton's approach was primarily cinemetical, Leibniz' was geometrical; he thought in terms of the "characteristic triangle" (dx, dy, ds), which had already appeared in several other writings, notably in Pascal and in Barrow's "Geometrical Lectures" of 1670.[1] The first publication of Leibniz' form of calculus occurred in 1684 in a six page article in the "Acta Eruditorum," a mathematical periodical which he had founded in 1682. The paper had the characteristic title, "Nova methodus pro maximis et minimis, itemque tangentibus, quae nec fractas nec irrationales quantitates moratur, et singulare pro illi calculi genus."[2] It was a barren and obscure account, but it contained our symbols dx, dy and the rules of differentiation, including $d(uv) = udv + vdu$ and the differential for the quotient, with the condition $dy = 0$ for extreme values and $d^2y = 0$ for points of inflexion. This paper was followed in 1686 by another with the rules of the integral calculus, containing the $\int$ symbol. It expressed the equation of the cycloid as

$$y = \sqrt{2x - x^2} + \int \frac{dx}{\sqrt{2x - x^2}}$$

An extremely fertile period of mathematical productivity began with the publication of these papers. Leibniz was joined after 1687 by the Bernoulli brothers who eagerly absorbed his methods. Before 1700 these men had found most of our undergraduate calculus, together with important sections of more advanced

[1]The term "triangulum characteristicum" seems to have first been used by Leibniz, who found it by reading Pascal's *Traité des sinus du quart de cercle*, part of his Dettonville letters (1658). It had already occurred in Snellius' *Tiphys Batavus* (1624) pp. 22-25.

[2]"A new method for maxima and minima, as well as tangents, which is not obstructed by fractional and irrational quantities, and a curious type of calculus for it."

fields, including the solution of some problems in the calculus of variations. By 1696 the first textbook on calculus appeared, written by the Marquis de l'Hospital, a pupil of Johann Bernoulli, who published his teacher's lectures on the differential calculus in the "Analyse des infiniment petits." This book contains the so-called "rule of l'Hospital" of finding the limiting value of a fraction whose two terms both tend toward zero.

Our notation of the calculus is due to Leibniz, even the names "calculus differentialis" and "calculus integralis."[1] Because of his influence the sign = is used for equality and the · for multiplication. The terms "function" and "coordinates" are due to Leibniz, as well as the playful term "osculating." The series

$$\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots$$

are named after Leibniz, though he has not the priority of the discovery. (This seems to go to James Gregory, a Scotch mathematician, who also tried to prove that the quadrature of the circle with compass and ruler is impossible.)

Leibniz' explanation of the foundations of the calculus suffered from the same vagueness as Newton's. Sometimes his *dx*, *dy* were finite quantities, sometimes quan-

[1] Leibniz suggested the name "calculus summatorius" first, but in 1696 Leibniz and Johann Bernoulli agreed on the name "calculus integralis." Modern analysis has returned to Leibniz' early terminology. See further: F. Cajori, *Leibniz, the Master Builder of Mathematical Notations*, Isis 7 (1925) pp. 412-429.

I.

NOVA METHODUS PRO MAXIMIS ET MINIMIS, ITEMQUE TANGENTIBUS, QUAE NEC FRACTAS NEC IRRATIONALES QUANTITATES MORATUR, ET SINGULARE PRO ILLIS CALCULI GENUS*).

Sit (fig. 111) axis AX, et curvae plures, ut VV, WW, YY, ZZ, quarum ordinatae ad axem normales, VX, WX, YX, ZX, quae vocentur respective v, w, y, x, et ipsa AX, abscissa ab axe, vocetur x. Tangentes sint VB, WC, YD, ZE, axi occurrentes respective in punctis B, C, D, E. Jam recta aliqua pro arbitrio assumta vocetur dx, et recta, quae sit ad dx, ut v (vel w, vel y, vel z) est ad XB (vel XC, vel XD, vel XE) vocetur dv (vel dw, vel dy, vel dz) sive differentia ipsarum v (vel ipsarum w, vel y, vel z). His positis, calculi regulae erunt tales.

Sit a quantitas data constans, erit da aequalis 0, et $d\overline{ax}$ erit aequalis adx. Si sit y aequ. v (seu ordinata quaevis curvae YY aequalis cuivis ordinatae respondenti curvae VV) erit dy aequ. dv. Jam *Additio et Subtractio*: si sit z — y + w + x aequ. v, erit $d\overline{z - y + w + x}$ seu dv aequ. dz — dy + dw + dx. *Multiplicatio*: $d\overline{xv}$ aequ. xdv + vdx, seu posito y aequ. xv, fiet dy aequ. xdv + vdx. In arbitrio enim est vel formulam, ut xv, vel compendio pro ea literam, ut y, adhibere. Notandum, et x et dx eodem modo in hoc calculo tractari, ut y et dy, vel aliam literam indeterminatam cum sua differentiali. Notandum etiam, non dari semper regressum a differentiali Aequatione, nisi cum quadam cautione, de quo alibi. Porro *Divisio*: $d\frac{v}{y}$ vel (posito z aequ. $\frac{v}{y}$) dz aequ. $\frac{\pm v dy \mp y dv}{yy}$.

Quoad *Signa* hoc probe notandum, cum in calculo pro litera substituitur simpliciter ejus differentialis, servari quidem eadem signa, et pro + z scribi + dz, pro — z scribi — dz, ut ex addi-

*) Act. Erud. Lips. an. 1684.

LEIBNIZ'S FIRST PAPER ON THE CALCULUS (As reprinted by C. I. Gerhardt, 1858)

tities less than any assignable quantity and yet not zero. In the absence of rigorous definitions he presented analogies pointing to the relation of the radius of the earth to the distance of the fixed stars. He varied his modes of approach to questions concerning the infinite; in one of his letters (to Foucher, 1693) he accepted the existence of the actually infinite to overcome Zeno's difficulties and praised Grégoire de Saint Vincent who had computed the place where Achilles meets the tortoise. And just as Newton's vagueness provoked the criticism of Berkeley, so Leibniz' vagueness provoked the opposition of Bernard Nieuwentijt, burgomaster of Purmerend near Amsterdam (1694). Both Berkeley's and Nieuwentijt's criticism had their justification, but they were entirely negative. They were unable to supply a rigorous foundation to the calculus, but inspired further constructive work.

Literature.

The collected works of Descartes, Pascal, Huygens (partial), and Fermat are available in modern editions, those of Leibniz in a somewhat older edition (incomplete).

For the discovery of the calculus see:

C. B. Boyer, *The Concepts of the Calculus* (London, 1939), esp. Ch. IV and V. This book has a large bibliography.

On the historical—technical background:

H. Grossmann, *Die gesellschaftlichen Grundlagen der mechanistischen Philosophie und die Manifaktur*, Zeitschrift für Sozialforschung 4 (1935) pp. 161-231.

On the leading mathematicians:

The Mathematical Works of John Wallis D.D., T.R.S. (London, 1938).

A. Prag, *John Wallis, Zur Ideengeschichte der Mathematik im 17. Jahrhundert.* Quellen and Studien B[1] (1930) pp. 381-412.
Also see T. P. Nunn, Math. Gaz. 5 (1910-11).

Barrow, *Geometrical Lectures*, transl. and ed. by J. M. Child (Chicago, 1916).

A. E. Bell, *Christiaan Huygens and the Development of Science in the Seventeenth Century* (London, 1948).

L. T. More, *Isaac Newton. A Biography* (N. Y., London, 1934).

Principia, English ed. by F. Cajori (Un. of California, 1934).

Collections of papers on Newton have been published by the History of Science Soc. (Baltimore, 1928); by the Mathem. Assoc. (London, 1927); and by the Royal Soc. (Cambridge, 1947).

J. M. Child, *The Early Mathematical Manuscripts of Leibniz*, transl. from the Latin texts (Chicago, 1920).

Johann Kepler. A Tercentenary Commemoration of His Life and Works (Baltimore, 1931).

G. Milhaud, *Descartes savant* (Paris, 1921).

J. E. Hofmann, *Die Entwicklungsgeschichte der Leibnizschen Mathematik während des Aufenthaltes in Paris (1672–1676)* (München, 1949).

H. Bosmans (see p. 122) has papers on:

Tacquet (Isis 9, 1927-28, pp. 66-83), Stevin (Mathesis 37, 1923; Ann. Soc. Sc. Bruxelles 37, 1913, pp. 171-199; Biographie nationale de Belgique), Della Faille (Mathesis 41, 1927, pp. 5-11), De Saint Vincent (Mathesis 38, 1924, pp. 250-256).

O. Toeplitz, *Die Entwicklung der Infinitesimalrechnung I* (Berlin, 1949).

On Stevin also:

G. Sarton, *Simon Stevin of Bruges*, Isis 21 (1934) pp. 241-303.

E. J. Dÿksterhuis, *Simon Stevin* ('s-Gravenhage, 1943).

See further:

Paul Tannery, *Notions Historiques*, in J. Tannery, Notions de mathématiques (Paris, 1903) pp. 324-348.

CHAPTER VII

The Eighteenth Century

1. Mathematical productivity in the Eighteenth Century concentrated on the calculus and its application to mechanics. The major figures can be arranged in a kind of pedigree to indicate their intellectual kinship:

Leibniz—(1646-1716)

The brothers Bernoulli: Jakob (1654-1705), Johann (1667-1748)

Euler—(1707-1783)

Lagrange—(1736-1813)

Laplace—(1749-1827)

Closely related to the work of these men was the activity of a group of French mathematicians, notably Clairaut, D'Alembert, and Maupertuis, who were again connected with the philosophers of the Enlightenment. To them must be added the Swiss mathematicians Lambert and Daniel Bernoulli. Scientific activity usually centered around Academies, of which those at Paris, Berlin, and St. Petersburg were outstanding. University teaching played a minor role or no role at all. It was a period in which some of the leading European countries were ruled by what has euphemistically been called enlightened despots: Frederick the Great, Catherine the Great, perhaps also Louis XV. Part of these despots' claim to glory was their delight in having learned men around. This delight was a type of intellectual snobbery, tempered by some understanding of the important role which natural science and applied mathe-

matics were taking in improving manufactures and increasing the efficiency of the military. It is said, for instance, that the excellence of the French navy was due to the fact that in the construction of frigates and of ships of the line the master shipbuilders were partly led by mathematical theory. Euler's works abound in applications to questions of importance to army and navy. Astronomy continued to play its outstanding role as foster-mother to mathematical research under royal and imperial protection.

2. Basle in Switzerland, a free empire city since 1263, had long been a center of learning. In the days of Erasmus its university was already a great center. The arts and sciences flourished in Basle, as in the cities of Holland, under the rule of a merchant patriciate. To this Basle patriciate belonged the merchant family of the Bernoullis, who had come from Antwerp in the previous century after that city had been conquered by the Spanish. From the late Seventeenth Century to the present time this family in every generation has produced scientists. Indeed it is difficult to find in the whole history of science a family with a more distinguished record.

This record begins with two mathematicians, Jakob (James, Jacques) and Johann (John, Jean) Bernoulli. Jakob studied theology, Johann studied medicine; but when Leibniz' papers in the "Acta eruditorum" appeared both men decided to become mathematicians. They became the first important pupils of Leibniz. In 1687 Jakob accepted the chair of mathematics at Basle University where he taught until his death in

1705. In 1697 Johann became professor at Groningen; on his brother's death he succeeded him in his chair at Basle where he stayed for forty-three more years.

Jakob began his correspondence with Leibniz in 1687. Then in constant exchange of ideas with Leibniz and with each other—often in bitter rivalry among themselves—the two brothers began to discover the treasures contained in Leibniz' pioneering venture. The list of their results is long and contains not only much of the material now contained in our elementary texts on differential and integral calculus but also in the integration of many ordinary differential equations. Among Jakob's contributions are the use of polar coordinates, the study of the catenary (already discussed by Huygens and others), the lemniscate (1694), and the logarithmic spiral. In 1690 he found the so-called isochrone, proposed by Leibniz in 1687 as the curve along which a body falls with uniform velocity; it appeared to be a semi-cubic parabola. Jakob also discussed isoperimetric figures (1701), which led to a problem in the calculus of variations. The logarithmic spiral, which has a way of reproducing itself under various transformations (its evolute is a logarithmic spiral and so are both the pedal curve and the caustic with respect to the pole), was such a delight to Jakob that he willed that the curve be engraved on his tombstone with the inscription "eadem mutata resurgo."[1]

Jakob Bernoulli was also one of the early students of the theory of probabilities, on which subject he wrote the "Ars conjectandi," published posthumously in

[1] "I arise the same though changed." The spiral on the gravestone, however, looks like an Archimedian spiral.

1713. In the first part of this book Huygens' tract on games of chance is reprinted; the other parts deal with permutations and combinations and come to a climax in the "theorem of Bernoulli" on binomial distributions. "Bernoulli's numbers" appear in this book in a discussion of Pascal's triangle.

3. Johann Bernoulli's work was closely related to that of his older brother, and it is not always easy to discriminate between the results of these two men. Johann is often considered the inventor of the calculus of variations because of his contribution to the problem of the brachystochrone. This is the curve of quickest descent for a mass point moving between two points in a gravitational field, a curve studied by Leibniz and the Bernoullis in 1697 and the following years. At this time they found the equation of the geodesics on a surface.[1] The answer to the problem of the brachystochrone is the cycloid. This curve also solves the problem of the tautochrone, the curve along which a mass point in a gravitational field reaches the lowest point in a time independent of its starting point. Huygens discovered this property of the cycloid and used it in constructing tautochronous pendulum clocks (1673) in which the period is independent of the amplitude.

Among the other Bernoullis who have influenced the course of mathematics are two sons of Johann: Nicolaus,

[1] Newton in a scholium of the "Principia" (II, Prop. 35) had already discussed the solid of revolution moving in a liquid with least resistance. He published no proof of his contention.

and above all Daniel.[2] Nicolaus was called to St. Petersburg, founded only a few years before by Czar Peter the Great; he stayed there for a short period. The problem in the theory of probability which he proposed from that city is known as the "problem" (or, more dramatically, the "paradox") of St. Petersburg. This son of Johann died young but the other, Daniel, lived to a ripe age. Until 1777 he was professor at the University of Basle. Daniel's prolific activity was mainly devoted to astronomy, physics, and hydrodynamics. His "Hydrodynamica" appeared in 1738 and one of its theorems on hydraulic pressure carries his name. In the same year he established the kinetic theory of gasses; with D'Alembert and Euler he studied the theory of the vibrating string. Where his father and uncle developed the theory of ordinary differential equations, Daniel pioneered in partial differential equations.

4. Also from Basle came the most productive mathematician of the Eighteenth Century—if not of all times—Leonard Euler. His father studied mathematics under Jakob Bernoulli and Leonard studied it under Johann.

[2] Nicolaus Bernoulli

Jakob (1654-1705) "Ars conjectandi"

Johann (1667-1748) "brachystochrone"

Nicolaus (1695-1726) "St. Petersburg problem"

Daniel (1700-1784) "hydrodynamics"

When in 1725 Johann's son Nicolaus travelled to St. Petersburg, young Euler followed him and stayed at the Academy until 1741. From 1741 to 1766 Euler was at the Berlin Academy under the special tutelage of Frederick the Great; from 1766-1783 he was again at St. Petersburg, now under the egis of the Empress Catherine. He married twice and had thirteen children. The life of this Eighteenth Century academician was almost exclusively devoted to work in the different fields of pure and applied mathematics. Although he lost one eye in 1735 and the other eye in 1766, nothing could interrupt his enormous productivity. The blind Euler, aided by a phenomenal memory, continued to dictate his discoveries. During his life 530 books and papers appeared; at his death he left many manuscripts, which were published by the St. Petersburg academy during the next forty-seven years. (This brings the number of his works to 771, but research by Gustav Eneström has completed the list to 886.)

Euler made signal contributions in every field of mathematics which existed in his day. He published his results not only in articles of varied length but also in an impressive number of large textbooks which ordered and codified the material assembled during the ages. In several fields Euler's presentation has been almost final. An example is our present trigonometry with its conception of trigonometric values as ratios and its usual notation, which dates from Euler's "Introductio in analysin infinitorum" (1748). The tremendous prestige of his textbooks settled for ever many moot questions of notation in algebra and calculus; Lagrange,

Laplace, and Gauss knew and followed Euler in all their works.

The "Introductio" of 1748 covers in its two volumes a wide variety of subjects. It contains an exposition of infinite series including those for e^x, $\sin x$, and $\cos x$, and presents the relation $e^{ix} = \cos x + i \sin x$ (already discovered by Johann Bernoulli and others in different forms). Curves and surfaces are so freely investigated with the aid of their equations that we may consider the "Introductio" the first text on analytic geometry. We also find here an algebraic theory of elimination. To the most exciting parts of this book belong the chapter on the Zeta function and its relation to the prime number theory, as well as the chapter on partitio numerorum.[1]

Another great and rich textbook was Euler's "Institutiones calculi differentialis" (1755), followed by three volumes of "Institutiones calculi integralis" (1768-1774). Here we find not only our elementary differential and integral calculus but also a theory of differential equations, Taylor's theorem with many applications, Euler's "summation" formula, and the Eulerian integrals Γ and B. The section on differential equations with its distinction between "linear", "exact", and "homogeneous" equations is still the model of our elementary texts on this subject.

Euler's "Mechanica, sive motus scientia analytice exposita" (1736) was the first textbook in which Newton's dynamics of the mass point was developed with analytical methods. It was followed by the "Theoria motus corporum solidorum seu rigidorum" (1765) in

[1]See the preface to the "Introductio" by A. Speiser: Euler, *Opera* I, 9 (1945).

LEONARD EULER (1707–1783)
From the portrait by A. Lorgna

which the mechanics of solid bodies was similarly treated. This textbook contains the "Eulerian" equations for a body rotating about a point. The "Vollstaendige Anleitung zur Algebra" (1770)—written in German and dictated to a servant—has been the model of many later texts on algebra. It leads up to the theory of cubic and biquadratic equations.

In 1744 appeared Euler's "Methodus inveniendi lineas curvas maximi minimi ve proprietate gaudentes." This was the first exposition of the calculus of variations; it contained "Euler's equations" with many applications, including the discovery that catenoid and right helicoid are minimal surfaces. Many other results of Euler can be found in his smaller papers which contain many a gem, little known even today. To the better known discoveries belong the theorem connecting the number of vertices (V), edges (E), and faces (F) of a closed polyhedron ($V + F - E = 2$)[1]; the line of Euler in the triangle; the curves of constant width (Euler called them orbiform curves); and the constant of Euler

$$\lim_{n\to\infty}\left(\frac{1}{1}+\frac{1}{2}+\cdots+\frac{1}{n}-\log n\right) = \cdot 577216\cdots$$

Several papers are devoted to mathematical recreations (the seven bridges of Königsberg, the knight's jump in chess). Euler's contributions to the theory of numbers alone would have given him a niche in the hall of fame, to his discoveries in this field belongs the law of quadratic reciprocity.

A large amount of Euler's activity was devoted to astronomy, where the lunar theory, an important sec-

[1]Already known to Descartes.

Secantes autem et cosecantes ex tangentibus per solam subtractionem inveniuntur; est enim

$$\operatorname{cosec.} z = \cot. \frac{1}{2} z - \cot. z$$

et hinc

$$\sec. z = \cot. \left(45^\circ - \frac{1}{2} z\right) - \operatorname{tang.} z.$$

Ex his ergo luculenter perspicitur, quomodo canones sinuum construi potuerint.

138. Ponatur denuo in formulis § 133 arcus z infinite parvus et sit n numerus infinite magnus i, ut iz obtineat valorem finitum v. Erit ergo $nz = v$ et $z = \frac{v}{i}$, unde $\sin. z = \frac{v}{i}$ et $\cos. z = 1$; his substitutis fit

$$\cos. v = \frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i + \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2}$$

atque

$$\sin. v = \frac{\left(1 + \frac{v\sqrt{-1}}{i}\right)^i - \left(1 - \frac{v\sqrt{-1}}{i}\right)^i}{2\sqrt{-1}}.$$

In capite autem praecedente vidimus esse

$$\left(1 + \frac{z}{i}\right)^i = e^z$$

denotante e basin logarithmorum hyperbolicorum; scripto ergo pro z partim $+v\sqrt{-1}$ partim $-v\sqrt{-1}$ erit

$$\cos. v = \frac{e^{+v\sqrt{-1}} + e^{-v\sqrt{-1}}}{2}$$

et

$$\sin. v = \frac{e^{+v\sqrt{-1}} - e^{-v\sqrt{-1}}}{2\sqrt{-1}}.$$

Ex quibus intelligitur, quomodo quantitates exponentiales imaginariae ad sinus et cosinus arcuum realium reducantur.[1] Erit vero

1) Has celeberrimas formulas, quas ab inventore *Formulas Eulerianas* nominare solemus, Eulerus distincte primum exposuit in Commentatione 61 (indicis Enestroemiani): *De summis*

19*

THE PAGES IN EULER'S *Introductio* WHERE $e^{ix} = \cos x + i \sin x$ IS INTRODUCED

$$e^{+v\sqrt{-1}} = \cos. v + \sqrt{-1} \cdot \sin. v$$

et

$$e^{-v\sqrt{-1}} = \cos. v - \sqrt{-1} \cdot \sin. v.$$

139. Sit iam in iisdem formulis § 133 n numerus infinite parvus seu $n = \frac{1}{i}$ existente i numero infinite magno; erit

$$\cos. nz = \cos. \frac{z}{i} = 1 \quad \text{et} \quad \sin. nz = \sin. \frac{z}{i} = \frac{z}{i};$$

arcus enim evanescentis $\frac{z}{i}$ sinus est ipsi aequalis, cosinus vero $= 1$. His positis habebitur

$$1 = \frac{(\cos. z + \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}} + (\cos. z - \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}}}{2}$$

et

$$\frac{z}{i} = \frac{(\cos. z + \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}} - (\cos. z - \sqrt{-1} \cdot \sin. z)^{\frac{1}{i}}}{2\sqrt{-1}}.$$

Sumendis autem logarithmis hyperbolicis supra (§ 125) ostendimus esse

$$l(1 + x) = i(1 + x)^{\frac{1}{i}} - i \quad \text{seu} \quad y^{\frac{1}{i}} = 1 + \frac{1}{i} l y$$

posito y loco $1 + x$. Nunc igitur posito loco y partim $\cos. z + \sqrt{-1} \cdot \sin. z$ partim $\cos. z - \sqrt{-1} \cdot \sin. z$ prodibit

seriierum reciprocarum ex potestatibus numerorum naturalium ortarum, Miscellanea Berolin. 7, 1743, p. 172; *Leonhardi Euleri Opera omnia*, series I, vol. 14. Iam antea quidem cum amico Chr. Goldbach (1690—1764) formulas huc pertinentes, partim speciales partim generaliores, communicaverat. Sic in epistola d. 9. Dec. 1741 scripta invenitur haec formula

$$\frac{2^{+\sqrt{-1}} + 2^{-\sqrt{-1}}}{2} = \text{Cos. Arc. } l2$$

et in epistola d. 8. Maii 1742 scripta haec

$$a^{p\sqrt{-1}} + a^{-p\sqrt{-1}} = 2 \text{ Cos. Arc. } pla.$$

Vide *Correspondance math. et phys. publiée par P. H. Fuss*, St.-Pétersbourg 1843, t. I, p. 110 et 123; *Leonhardi Euleri Opera omnia*, series III. Confer etiam Commentationem 170 nota 1 p. 35 laudatam, imprimis § 90 et 91. A. K.

tion of the three-body problem, received his special attention. The "Theoria motus planetarum et cometarum" (1774) is a treatise on celestial mechanics. Related to this work was Euler's study of the attraction of ellipsoids (1738).

There are books by Euler on hydraulics, on ship construction, on artillery. In 1769-71 there appeared three tomes of a "Dioptrica" with a theory of the passage of rays through a system of lenses. In 1739 appeared his new theory of music, of which it has been said that it was too musical for mathematicians and too mathematical for musicians. Euler's philosophical exposition of the most important problems of natural science in his "Letters to a German Princess" (written 1760–61) remained a model of popularization.

The enormous fertility of Euler has been a continuous source of surprise and admiration for everyone who has attempted to study his work, a task not so difficult as it seems, since Euler's Latin is very simple and his notation is almost modern—or perhaps we should better say that our notation is almost Euler's! A long list can be made of the known discoveries of which Euler possesses priority, and another a list of ideas which are still worth elaborating. Great mathematicians have always appreciated their indebtedness to Euler. "Lisez Euler," Laplace used to say to younger mathematicians, "lisez Euler, c'est notre maitre à tous." And Gauss, more ponderously, expressed himself: "The study of Euler's works will remain the best school for the different fields of mathematics and nothing else can replace it." Riemann knew Euler's works well and some of his most profound works have an Eulerian touch.

Publishers might do worse than offer translations of some of Euler's works together with modern commentaries.

5. It is instructive to point out not only some of Euler's contributions to science but also some of his weaknesses. Infinite processes were still carelessly handled in the Eighteenth Century and much of the work of the leading mathematicians of that period impresses us as wildly enthusiastic experimentation. There was experimentation with infinite series, with infinite products, with integration, with the use of symbols such as 0, ∞, $\sqrt{-1}$. If many of Euler's conclusions can be accepted today, there are others concerning which we have reservations. We accept, for instance, Euler's statement that $\log n$ has an infinity of values which are all complex numbers, except when n is positive, when one of the values is real. Euler came to this conclusion in a letter to D'Alembert (1747) who had claimed that $\log(-1) = 0$. But we cannot always follow Euler when he writes that $1 - 3 + 5 - 7 + \cdots = 0$, or when he concludes from

$$n + n^2 + \cdots = \frac{n}{1-n}$$

$$\text{and } 1 + \frac{1}{n} + \frac{1}{n^2} + \cdots = \frac{n}{n-1}$$

that

$$\cdots + \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + \cdots = 0.$$

Yet we must be careful and not criticize Euler too hastily for his way of manipulating divergent series; he simply did not always use some of our present tests of convergence or divergence as a criterium for the validity of his series. Much of his supposedly indiscriminate work with series has been given a strictly rigorous sense by modern mathematicians.

We cannot, however, be enthusiastic about Euler's way of basing the calculus on the introduction of zeros of different orders. An infinitesimally small quantity, wrote Euler in his "Differential Calculus" of 1755, is truly zero, $a \pm ndx = a,$[1] $dx \pm (dx)^{n+1} = dx$, and $a\sqrt{dx} + Cdx = a\sqrt{dx}$

> "Therefore there exist infinite orders of infinitely small quantities, which, though they all = 0, still have to be well distinguished among themselves, if we look at their mutual relation, which is explained by a geometrical ratio."

The whole question of the foundation of the calculus remained a subject of debate, and so did all questions relating to infinite processes. The "mystical" period in the foundation of the calculus (to use a term suggested by Karl Marx) itself provoked a mysticism which occasionally went far beyond that of the founding fathers. Guido Grandi, a monk and professor at Pisa known for his study of rosaces ($r = \sin n\,\Theta$) and other curves which resemble flowers, considered the formula

[1]This formula reminds us of a statement ascribed to Zeno by Simplicius: "That which, being added to another, does not make it greater, and being taken away from another, does not make it less, is nothing".

$$\begin{aligned}\frac{1}{2} &= 1 - 1 + 1 - 1 + 1 - \cdots \\ &= (1 - 1) + (1 - 1) + (1 - 1) + \cdots \\ &= 0 + 0 + 0 + \cdots\end{aligned}$$

as the symbol for Creation from Nothing. He obtained the result 1/2 by considering the case of a father who bequeathes a gem to his two sons who each may keep the bauble one year in alternation. It then belongs to each son for one half.

Euler's foundation of the calculus may have had its weakness, but he expressed his point of view without vagueness. D'Alembert, in some articles of the "Encyclopédie," attempted to find this foundation by other means. Newton had used the term "prime and ultimate ratio" for the "fluxion," as the first or last ratio of two quantities just springing into being. D'Alembert replaced this notion by the conception of a *limit.* One quantity he called the limit of another when the second approaches the first nearer than by any given quantity. "The differentiation of equations consists simply in finding the limits of the ratio of finite differences of two variables included in the equation." This was a great step ahead, as was D'Alembert's conception of infinites of different orders. However, his contemporaries were not easily convinced of the importance of the new step and when D'Alembert said that the secant becomes the tangent when the two points of intersection are one, it was felt that he had not overcome the difficulties inherent in Zeno's paradoxes. After all, does a variable quantity reach its limit? or does it never reach it?

We have already referred to Bishop Berkeley's criticism of Newton's fluxions. George Berkeley, first dean of Derry, after 1734 Bishop of Cloyne in S. Ireland—and from 1729 to 1731 a resident of Newport R. I.—is primarily known for his extreme idealism ("esse est percipi"). He resented the support which Newtonian science gave to materialism and he attacked the theory of fluxions in the "Analyst" of 1734. He derided the infinitesimals as "ghosts of departed quantities"; if x receives an increment o, then the increment of x^n, divided by o, is $nx^{n-1} + \frac{n(n-1)}{1.2} x^{n-2} o + \cdots$. This is obtained by supposing o different from zero. The fluxion of x^n, nx^{n-1}, however, is obtained by taking o as equal to zero, when the hypothesis is suddenly shifted, since o was supposed to be different from zero. This was the "manifest sophism," which Berkeley discovered in the calculus, and he believed that its correct results were obtained by a compensation of errors. Fluxions were logically unaccountable. "But he who can digest a second or third Fluxion, a second or third Difference," Berkeley exclaimed to the "infidel mathematician" whom he addressed (Halley), "need not, methinks, be squeamish about any Point in Divinity." It has not been the only case in which a critical difficulty in a science has been used to strengthen an idealist philosophy.

John Landen, a self-taught British mathematician whose name is preserved in the theory of elliptic integrals, tried to overcome the basic difficulties in the calculus in his own way. In the "Residual analysis" (1764) he met Berkeley's criticism by avoiding infinitesimals altogether; the derivative of x^3, for instance, was found by changing x into x_1, after which

$$\frac{x_1^3 - x^3}{x_1 - x} = x_1^2 + xx_1 + x^2$$

becomes $3x^2$ when $x = x_1$. Since this procedure involves infinite series when the functions are more complicated, Landen's method has some affinity to the later "algebraic" method of Lagrange.

6. Although Euler was incontestably the leading mathematician of this period, France continued to produce work of great originality. Here, more than in any other country, mathematics was conceived as the science which was to bring Newton's theory to greater perfection. The theory of universal gravitation had great attraction for the philosophers of the Enlightenment, who used it as a weapon in their struggle against the remnants of feudalism. The Catholic Church had placed Descartes on the index in 1664, but by 1700 his theories had become fashionable even in conservative circles. The question of Newtonianism versus Cartesianism became for a while a topic of the greatest interest not only in learned circles but also in the salons. Voltaire's "Lettres sur les Anglais" (1734) did much to introduce Newton to the French reading public; Voltaire's friend Mme. Du Châtelet even translated the "Principia" into French (1759). A particular point of contention between the two schools was the figure of the earth. In the cosmogony favored by the Cartesians the earth elongated at the poles; Newton's theory required that it be flattened. The Cartesian astronomers Cassini (Jean

Dominique the father and Jacques the son; the father known in geometry because of "Cassini's oval," 1680) had measured an arc of the meridian in France between 1700 and 1720 and vindicated the Cartesian conclusion. A controversy arose in which many mathematicians participated. An expedition was sent in 1735 to Peru, followed in 1736-37 by one under the direction of Pierre De Maupertuis to the Tornea in Lapland in order to measure a degree of longitude. The result of both expeditions was a triumph for Newton's theory, as well as for Maupertuis himself. The now famous "grand aplatisseur" ("grand flattener") became president of the Berlin Academy and basked for many years in the sun of his fame at the court of Frederick the Great. This lasted until 1750 when he entered into a spirited controversy with the Swiss mathematician Samuel König concerning the principle of least action in mechanics, perhaps already indicated by Leibniz. Maupertuis was looking, as Fermat had done before and Einstein has done after him, for some general principle by which the laws of the universe could be unified. Maupertuis' formulation was not clear, but he defined as his "action" the quantity mvs (m = mass, v = velocity, s = distance); he combined with it a proof of the existence of God. The controversy was brought to a climax when Voltaire lampooned the unhappy president in the "Diatribe du docteur Akakia, Médecin du pape" (1752). Neither the king's support nor Euler's defense could bring succor to Maupertuis' sunken spirits, and the deflated mathematician died not long afterwards in Basle in the home of the Bernoullis.

Euler restated the principle of least action in the

form that $\int mvds$ must be a minimum; moreover he did not indulge in Maupertuis' metaphysics. This placed the principle on a sound basis, where it was used by Lagrange[1] and later by Hamilton. The use of the "Hamiltonian" in modern mathematical physics illustrates the fundamental character of Euler's contribution to the Maupertuis-König controversy.

Among the mathematicians who went with Maupertuis to Lapland was Alexis Claude Clairaut. Clairaut, at eighteen years of age, had published the "Recherches sur les courbes à double courbure," a first attempt to deal with the analytical and differential geometry of space curves. On his return from Lapland Clairaut published his "Théorie de la figure de la terre" (1743), a standard work on the equilibrium of fluids and the attraction of ellipsoids of revolution. Laplace could only improve on it in minor details. Among its many results is the condition that a differential $Mdx + Ndy$ be exact. This book was followed by the "Théorie de la lune" (1752), which added material to Euler's theory of the moon's motion and the problem of three bodies in general. Clairaut also contributed to the theory of line integrals and of differential equations; one of the types which he considered is known as Clairaut's equation and offered one of the first known examples of a singular solution.

7. The intellectual opposition to the "ancien régime" centered after 1750 around the famous "Encyclopédie,"

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[1]See E. Mach, The Science of Mechanics (Chicago, 1893), p. 364.

28 vols. (1751-1772). The editor was Denis Diderot, under whose leadership the Encyclopedia presented a detailed philosophy of the Enlightenment. Diderot's knowledge of mathematics was not inconsiderable[1], but the leading mathematician of the Encyclopedists was Jean Le Rond D'Alembert, the natural son of an aristocratic lady, left as a foundling near the church of St. Jean Le Rond in Paris. His early brilliance facilitated his career; in 1754 he became "secrétaire perpetuel" of the French Academy and as such the most influential man of science in France. In 1743 appeared his "Traité de dynamique" which contains the method of reducing the dynamics of solid bodies to statics, known as "D'Alembert's principle." He continued to write on many applied subjects, especially on hydrodynamics,

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[1]There exists a widely quoted story about Diderot and Euler according to which Euler, in a public debate in St. Petersburg, succeeded in embarrassing the freethinking Diderot by claiming to possess an algebraic demonstration of the existence of God: "Sir, $(a + b^n)/n = x$; hence God exists, answer please!" This is a good example of a bad historical anecdote, since the value of an anecdote about an historical person lies in its faculty to illustrate certain aspects of his character; this particular anecdote serves to obscure both the character of Diderot and of Euler. Diderot knew his mathematics and had written on involutes and probability, and no reason exists to think that the thoughtful Euler would have behaved in the asinine way indicated. The story seems to have been made up by the English mathematician De Morgan (1806-73). See L. G. Krakeur—R. L. Krueger, Isis 31(1940) 431-432; also 33 (1941) 219-231. It is true that there was in the Eighteenth Century occasional talk about the possibility of an algebraic demonstration of the existence of God; Maupertuis indulged in one, see Voltaire's "Diatribe". Oeuvres 41 (1821 ed.) pp. 19, 30. See also B. Brown, Am. Math. Monthly 49 (1944).

JEAN LE ROND D'ALEMBERT (1717–1783)

aerodynamics, and the three-body problem. In 1747 appeared his theory of vibrating strings which made him together with Daniel Bernoulli the founder of the theory of partial differential equations. Where D'Alembert and Euler solved the equation $z_{tt} = k^2 z_{xx}$ by means of the expression $z = f(x + kt) + \phi(x - kt)$, Bernoulli solved it by means of a trigonometric series. There remained grave doubts concerning the nature of this solution; D'Alembert believed that the initial form of the string could only be given by a single analytical expression, while Euler thought that "any" continuous curve would do. Bernoulli believed, contrary to Euler, that his series solution was perfectly general. The full explanation of the problem had to wait until 1824 when Fourier removed the doubts concerning the validity of a trigonometric series as the representation of "any" function.

D'Alembert was a facile writer on many subjects, including even fundamental questions in mathematics. We mentioned his introduction of the limit conception. The "fundamental" theorem in algebra is sometimes called D'Alembert's theorem because of his attempt at proof (1746); and D'Alembert's "paradox" in the theory of probability shows that he also thought about the foundations of this theory—if not always very successfully.

The theory of probabilities made rapid advances in this period, mainly by further elaboration of the ideas of Fermat, Pascal, and Huygens. The "Ars conjectandi" was followed by several other texts, among them "The Doctrine of Chances" (1716) written by Abraham De Moivre, a French Huguenot who settled in London

after the revocation of the Edict of Nantes (1685) and earned a living by private tutoring. De Moivre's name is attached to a theorem in trigonometry, which in its present form $(\cos \varphi + i \sin \varphi)^n = \cos n\varphi + i \sin n\varphi$ appears first in Euler's "Introductio." In a paper published in 1733 he derived the normal probability function as an approximation to the binomial law and gave a formula equivalent to that of Stirling. James Stirling, an English mathematician of the Newtonian school, published his series in 1730.

The many lotteries and insurance companies which were organized in this period interested many mathematicians, including Euler, in the theory of probabilities. It led to attempts to apply the doctrine of chances to new fields. The Comte De Buffon, noted as the author of a natural history in 36 delightful volumes and the famous discourse on style (1753: "le style est de l'homme même") introduced in 1777 the first example of a geometrical probability. This was the so-called needle problem which has appealed to the imagination of many people because it allows the "experimental" determination of π by throwing a needle on a plane covered with parallel and equidistant lines and counting the number of times the needle hits a line.

To this period belong also the attempts to apply the theory of probability to man's judgment; for instance, by computing the chance that a tribunal can arrive at a true verdict if to each of the different witnesses a number can be given expressing the chance that he will speak the truth. This curious "probabilité des jugements," with its distinct flavor of Enlightenment philosophy, was prominent in the work of the Marquis De

Condorcet; it reappeared in Laplace and even in Poisson (1837).

8. De Moivre, Stirling, and Landen were good representatives of English Eighteenth Century mathematicians. We must report on a few more, though none of them reached the height of their continental colleagues. The tradition of the venerated Newton rested heavily upon English science and the clumsiness of his notation as compared to that of Leibniz made progress difficult. There were deep-lying social reasons why English mathematicians refused to be emancipated from Newtonian fluxional methods. England was in constant commercial wars with France and developed a feeling of intellectual superiority which was fostered not only by its victories in war and trade but also by the admiration in which the continental philosophers held its political system. England became the victim of its own supposed excellence. An analogy exists between the mathematics of Eighteenth Century England and of late Alexandrian antiquity. In both cases progress was technically impeded by an inadequate notation, but reasons for the self-satisfaction of the mathematicians were of a deeper lying social nature.

The leading English—or, rather, English speaking—mathematician of this period was Colin Maclaurin, professor at the University of Edinburgh, a disciple of Newton with whom he was personally acquainted. His study and extension of fluxional methods, of curves of second and higher order, and of the attraction of ellipsoids run parallel with contemporary efforts of Clairaut and Euler. Several of Maclaurin's theorems

occupy a place in our theory of plane curves and our projective geometry. In his "Geometria organica" (1720) we find the observation known as Cramer's paradox that a curve of the nth order is not always determined by $1/2\, n\, (n + 3)$ points, so that nine points may not uniquely determine a cubic while ten would be too many. Here we also find cinematical methods to describe plane curves of different degrees. Maclaurin's "Treatise of Fluxions" (2 vols., 1742)—written to defend Newton against Berkeley—is difficult to read because of the antiquated geometrical language, which is in sharp contrast with the ease of Euler's writing. Maclaurin used to obtain Archimedian rigor. The book contains Maclaurin's investigations on the attraction of ellipsoids of revolution and his theorem that two such ellipsoids, if they are confocal, attract a particle on the axis or in the equator with forces proportional to their volumes. In this "Treatise" Maclaurin also deals with the famous "series of Maclaurin."

This series, however, was no new discovery, since it had appeared in the "Methodus incrementorum" (1715) written by Brook Taylor, for a while secretary of the Royal Society. Maclaurin fully acknowledged his debt to Taylor. The series of Taylor is now always given in Lagrange's notation:

$$f(x + h) = f(x) + h\, f'(x) + \frac{h^2}{2!}\, f''(x) + \cdots ;$$

Taylor explicitly mentions the series for $x = 0$, which many college texts still insist on naming "Maclaurin's series." Taylor's derivation did not include convergence considerations, but Maclaurin made a beginning with

such considerations—he even had the so-called integral test for infinite series. The full importance of Taylor's series was not recognized until Euler applied it in his differential calculus (1755). Lagrange supplied it with the remainder and used it as the foundation of his theory of functions. Taylor himself used his series for the integration of some differential equations.

10. Joseph Louis Lagrange was born in Turin of Italian-French ancestry. At nineteen years of age he became professor of mathematics in the artillery school of Turin (1755). In 1766, when Euler left Berlin for St. Petersburg, Frederick the Great invited Lagrange to come to Berlin, accompanying his invitation with a modest message which said that "it is necessary that the greatest geometer of Europe should live near the greatest of kings." Lagrange stayed at Berlin until the death of Frederick (1786) after which he went to Paris. During the Revolution he assisted in reforming weights and measures; later he became professor, first at the École Normale (1795), then at the Ecole Polytechnique (1797).

To Lagrange's earliest works belong his contributions to the calculus of variations. Euler's memoir on this subject had appeared in 1755. Lagrange observed that Euler's method had "not all the simplicity which is desirable in a subject of pure analysis." The result was Lagrange's purely analytical calculus of variations (1760-61), which is not only full of original discoveries but also has the historical material well arranged and assimilated—something quite typical of all Lagrange's work. Lagrange immediately applied his theory to prob-

JOSEPH LOUIS LAGRANGE (1736–1813)

lems of dynamics, in which he made full use of Euler's formulation of the principle of least action, the result of the lamentable "Akakia" episode. Many of the essential ideas of the "Mécanique analytique" thus date back to Lagrange's Turin days. He also contributed to one of the standard problems of his day, the theory of the moon. He gave the first particular solutions of the three-body problem. The theorem of Lagrange states that it is possible to start three finite bodies in such a manner that their orbits are similar ellipses all described in the same time (1772). In 1767 appeared his memoir "Sur la résolution des équations numériques" in which he presented methods of separating the real roots of an algebraic equation and of approximating them by means of continued fractions. This was followed in 1770 by the "Réflexions sur la résolution algébrique des équations" which dealt with the fundamental question of why the methods useful to solve equations of degree $n \leqq 4$ are not successful for $n > 4$. This led Lagrange to rational functions of the roots and their behavior under the permutations of the roots; the procedure which not only stimulated Ruffini and Abel in their work on the case $n > 4$, but also led Galois to his theory of groups. Lagrange also made progress in the theory of numbers when he investigated quadratic residues and proved, among many other theorems, that every integer is the sum of four or less than four squares.

Lagrange devoted the second part of his life to the composition of his great works, the "Mécanique analytique" (1788), the "Théorie des fonctions analytiques" (1797), and its sequel, the "Leçons sur le calcul des fonctions" (1801). The two books on functions were

an attempt to give a solid foundation to the calculus by reducing it to algebra. Lagrange rejected the theory of limits as indicated by Newton and formulated by D'Alembert. He could not well understand what happened when $\Delta y/\Delta x$ reaches its limit. In the words of Lazare Carnot, the "organisateur de la victoire" in the French Revolution, who also worried about Newton's method of infinitesimals:

> "That method has the great inconvenience of considering quantities in the state in which they cease, so to speak, to be quantities; for though we can always well conceive the ratio of two quantities, as long as they remain finite, that ratio offers to the mind no clear and precise idea, as soon as its terms become, the one and the other, nothing at the same time"[1].

Lagrange's method was different from that of his predecessors. He started with Taylor's series, which he derived with their remainder, showing in a rather naive way that "any" function $f(x)$ could be developed in such a series with the aid of a purely algebraic process. Then the derivatives $f'(x)$, $f''(x)$, etc., were defined as the coefficients h, h^2, $\cdots$ in the Taylor expansion of $f(x + h)$ in terms of h. (The notation $f'(x)$, $f''(x)$ is due to Lagrange.)

Though this "algebraic" method of founding the calculus turned out to be unsatisfactory and though Lagrange gave insufficient attention to the convergence of the series, the abstract treatment of a function was a considerable step ahead. Here appeared a first "theory

[1] L. Carnot, *Réflexions sur la métaphysique du calcul infinitésimal* (5th ed., Paris, 1881), p. 147, quoted by F. Cajori, Am. Math. Monthly 22 (1915), p. 148.

of functions of a real variable" with applications to a large variety of problems in algebra and geometry.

Lagrange's "Mécanique analytique" is perhaps his most valuable work and still amply repays careful study. In his book, which appeared a hundred years after Newton's "Principia," the full power of the newly developed analysis was applied to the mechanics of points and of rigid bodies. The results of Euler, of D' Alembert, and of the other mathematicians of the Eighteenth Century were assimilated and further developed from a consistent point of view. Full use of Lagrange's own calculus of variations made the unification of the varied principles of statistics and dynamics possible—in statistics by the use of the principle of virtual velocities, in dynamics by the use of D'Alembert's principle. This led naturally to generalized coordinates and to the equation of motion in their "Lagrangian" form:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = F_i .$$

Newton's geometrical approach was now fully discarded; Lagrange's book was a triumph of pure analysis. The author went so far as to stress in the preface: "on ne trouvera point de figures dans cet ouvrage, seulement des opérations algébriques."[1] It characterized Lagrange as the first true analyst.

11. With Pierre Simon Laplace we reach the last of

[1] "No figures will be found in this work, only algebraic operations." The word "algebraic" instead of "analytic" is characteristic.

PIERRE SIMON LAPLACE (1749–1827)

the leading Eighteenth Century mathematicians. The son of a small Normandy proprietor, he attended classes at Beaumont and Caen, and through the aid of D'Alembert became professor of mathematics at the military school of Paris. He had several other teaching and administrative positions and took part during the Revolution in the organization of the École Normale as well as of the École Polytechnique. Napoleon bestowed many honors upon him, but so did Louis XVIII. In contrast to Monge and Carnot, Laplace easily shifted his political allegiances and with it all was somewhat of a snob; but this easy conscience enabled him to continue his purely mathematical activity despite all political changes in France.

The two great works of Laplace which unify not only his own investigations but all previous work in their respective subjects are the "Théorie analytique des probabilités" (1812) and the "Mécanique céleste" (5 vols., 1799-1825). Both monumental works were prefaced by extensive expositions in non-technical terms, the "Essai philosophique sur les probabilités" (1814) and the "Exposition du système du monde" (1796). This "exposition" contains the nebular hypothesis, independently proposed by Kant in 1755 (and even before Kant by Swedenborg in 1734). The "Mécanique céleste" itself was the culmination of the work of Newton, Clairaut, D'Alembert, Euler, Lagrange, and Laplace on the figure of the earth, the theory of the moon, the three body problem, and the perturbations of the planets, leading up to the momentous problem of the stability of the solar system. The name "Laplace's equation"

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

reminds us that potential theory is part of the "Mécanique céleste." (The equation itself already had been found by Euler in 1752 when he derived some of the principal equations of hydrodynamics.) Around this five tome opus cluster many anecdotes. Well-known is Laplace's supposed answer to Napoleon who tried to tease him by the remark that God was not mentioned in his book, "Sire, je n'avais pas besoin de cette hypothèse."[1] And Nathaniel Bowditch of Boston who translated four volumes of Laplace's work into English has remarked: "I never came across one of Laplace's 'Thus it plainly appears' without feeling sure that I have hours of hard work before me to fill up the chasm and find out and show how it plainly appears." Hamilton's mathematical career began by finding a mistake in Laplace's "Mécanique céleste." Green, reading Laplace, received the idea of a mathematical theory of electricity.

The "Essai philosophique sur les probabilités" is a very readable introduction to the theory of probabilities; it contains Laplace's "negative" definition of probabilities by postulating "equally likely events":

> "The theory of chance consists in the reduction of all events of the same kind to a certain number of equally likely cases, that are cases such that we are equally undecided about their existence, and to determine the number of cases which are favorable to the event of which we seek the probability."

Questions concerning probability appear, according

[1] "Sire, I did not need this hypothesis."

to Laplace, because we are partly ignorant and partly knowing. This led Laplace to his famous statement which summarizes the Eighteenth Century interpretation of mechanical materialism:

"An intelligence which, for a given instant, knew all the forces by which nature is animated and the respective position of the beings which compose it, and which besides was large enough to submit these data to analysis, would embrace in the same formula the motions of the largest bodies of the universe and those of the lightest atom: nothing would be uncertain to it, and the future as well as the past would be present to its eyes. Human mind offers a feeble sketch of this intelligence in the perfection which it has been able to give to Astronomy."

The standard text itself is so full of material that many later day discoveries in the theory of probabilities can already be found in Laplace.[1] The stately tome contains an extensive discussion of games of chance and of geometrical probabilities, of Bernoulli's theorem and of its relation to the normal integral, and of the theory of least squares invented by Legendre. The leading idea is the use of the "Fonctions génératrices," of which Laplace shows the power for the solution of difference equations. It is here that the "Laplace transform" is introduced, which later became the key to the Heaviside operational calculus. Laplace also rescued from oblivion and reformulated a theory sketched by Thomas Bayes, an obscure English clergyman, which was posthumously published in 1763-64. This theory became known as the theory of inverse probabilities.

[1]E. C. Molina, *The Theory of Probability: Some comments on Laplace's Théorie analytique*, Bull. Am. Math. Soc. 36 (1930) pp. 369-392.

JEAN ETIENNE MONTUCLA (1725–1799)

12. It is a curious fact that toward the end of the century some of the leading mathematicians expressed the feeling that the field of mathematics was somehow exhausted. The laborious efforts of Euler, Lagrange, D'Alembert, and others had already led to the most important theorems; the great standard texts had placed them, or would soon place them, in their proper setting; the few mathematicians of the next generation would only find minor problems to solve. "Ne vous semble-t-il pas que la haute géométrie va un peu à décadence?" wrote Lagrange to D'Alembert in 1772. "Elle n'a d' autre soutien que vous et M. Euler."[1] Lagrange even discontinued working in mathematics for a while. D' Alembert had little hope to give. Arago, in his "Eloge of Laplace" (1842) later expressed a sentiment which may help us to understand this feeling:

> "Five geometers—Clairaut, Euler, D'Alembert, Lagrange and Laplace—shared among them the world of which Newton had revealed the existence. They explored it in all directions, penetrated into regions believed inaccessible, pointed out countless phenomena in those regions which observation had not yet detected, and finally—and herein lies their imperishable glory—they brought within the domain of a single principle, a unique law, all that is most subtle and mysterious in the motions of the celestial bodies. Geometry also had the boldness to dispose of the future; when the centuries unroll themselves they will scrupulously ratify the decisions of science."

Arago's oratory points to the main source of this

[1] "Does it not seem to you that the sublime geometry tends to become a little decadent? She has no other support than you and Mr. Euler." "Geometry" in Eighteenth Century French is used for mathematics in general.

"fin de siècle" pessimism, which consisted of the tendency to identify the progress of mathematics too much with that of mechanics and astronomy. From the times of ancient Babylon until those of Euler and Laplace astronomy had guided and inspired the most sublime discoveries in mathematics; now this development seemed to have reached its climax. However, a new generation, inspired by the new perspectives opened by the French Revolution and the flowering of the natural sciences, was to show how unfounded this pessimism was. This great new impulse came only in part from France; it also came, as often in the history of civilization, from the periphery of the political and economical centers, in this case from Gauss in Göttingen.

Literature.

The works of Lagrange and Laplace are available in modern editions, those of Euler only in part.

J. H. Lambert, *Opera mathematica* I, Zürich, 1946.
Pp. ix-xxxi preface by A. Speiser.

F. Cajori, *A History of the Conception of Limits and Fluxions in Great Britain from Newton to Woodhouse* (London, 1919).

P. E. B. Jourdain, *The Principle of Least Action* (Chicago, 1913).

L. G. DuPasquier, *Léonard Euler et ses amis* (Paris, 1927).

H. Andoyer, *L'oeuvre scientifique de Laplace* (Paris, 1922).

G. Loria, *Nel secondo centenario della nascita di G. L. Lagrange*, Isis 28 (1938) pp. 366-375 (with full bibliography).

H. Auchter, *Brook Taylor der Mathematiker und Philosoph* (Marburg, 1937).

This book shows from data in Leibniz' manuscripts that Leibniz had the Taylor series by 1694.

H. G. Green—H. J. J. Winter, *John Landen, F. R. S. (1719-90), Mathematician*, Isis 35 (1944) pp. 6-10.

[Th. Bayes], Facsimile of Two Papers, with commentaries by E. C. Molina and W. E. Deming (Washington, D. C., 1940).

K. Pearson, *Laplace*, Biometrica 21 (1929) pp. 202-216.

CHAPTER VIII

The Nineteenth Century

1. The French Revolution and the Napoleonic period created extremely favorable conditions for the further growth of mathematics. The way was open for the Industrial Revolution on the continent of Europe. It stimulated the cultivation of the physical sciences; it created new social classes with a new outlook on life, interested in science and in technical education. The democratic ideas invaded academic life; criticism rose against antiquated forms of thinking; schools and universities had to be reformed and rejuvenated.

The new and turbulent mathematical productivity was not primarily due to the technical problems raised by the new industries. England, the heart of the industrial revolution, remained mathematically sterile for several decades. Mathematics progressed most healthily in France and somewhat later in Germany, countries in which the ideological break with the past was most sharply felt and where sweeping changes were made, or had to be made, to prepare the ground for the new capitalist economic and political structure. The new mathematical research gradually emancipated itself from the ancient tendency to see in mechanics and astronomy the final goal of the exact sciences. The pursuit of science as a whole became even more detached from the demands of economic life or of warfare. The specialist developed, interested in science for its own sake. The connection with practice was never

entirely broken but it often became obscured. A division between "pure" and "applied" mathematics accompanied the growth of specialization.[1]

The mathematicians of the Nineteenth Century were no longer found around royal courts or in the salons of the aristocracy. Their chief occupation consisted no longer in membership in a learned academy; they were usually employed by universities or technical schools and were teachers as well as investigators. The Bernoullis, Lagrange, and Laplace had done occasional teaching. Now the teaching responsibility increased; mathematics professors became educators and examiners of the youth. The internationalism of previous centuries tended to be undermined by the growing relationship between the scientists of each nation,

[1]The difference in approach found its classical expression in the remark by Jacobi on the opinions of Fourier, who still represented the utilitarian approach of the Eighteenth Century: "Il est vrai que Monsieur Fourier avait l'opinion que le but principal des mathématiques était l'utilité publique et l'explication des phénomènes naturels; mais un philosophe comme lui aurait dû savoir que le but unique de la science, c'est l'honneur de l'esprit humain, et que sous ce titre une question de nombre vaut autant qu'une question du système du monde." ("It is true that Mr. Fourier believed that the main end of mathematics was public usefulness and the explanation of the phenomena in nature, but such a philosopher as he was should have known that the sole end of science is the honor of the human mind, and that from this point of view a question concerning number is as important as a question concerning the system of the world.") In a letter to Legendre 1830, Werke I p. 454 Gauss represented a synthesis of both opinions; he freely applied mathematics to astronomy, physics, and geodesy, but considered at the same time mathematics the queen of the sciences, and arithmetic the queen of mathematics.

though international exchange of opinion did remain. Scientific Latin was gradually replaced by the national languages. Mathematicians began to work in specialized fields; and while Leibniz, Euler, D'Alembert can be described as "mathematicians" (as "géomètres" in the Eighteenth Century meaning of the word), we think of Cauchy as an analyst, of Cayley as an algebrist, of Steiner as a geometer (even a "pure" geometer) and of Cantor as a pioneer in point sets. The time was ripe for "mathematical physicists" followed by men learned in "mathematical statistics" or "mathematical logic". Specialization was only broken on the highest level of genius; and it was from the works of a Gauss, a Riemann, a Klein, a Poincaré that Nineteenth Century mathematics received its most powerful impetus.

2. On the dividing line between Eighteenth and Nineteenth Century mathematics towers the majestic figure of Carl Friedrich Gauss. He was born in the German city of Brunswick, the son of a day laborer. The duke of Brunswick gracefully recognized in young Gauss an infant prodigy and took charge of his education. The young genius studied from 1795-98 at Göttingen and in 1799 obtained his doctor's degree at Helmstädt. From 1807 until his death in 1855 he worked quietly and undisturbed as the director of the astronomical observatory and professor at the university of his alma mater. His comparative isolation, his grasp of "applied" as well as "pure" mathematics, his preoccupation with astronomy and his frequent use of Latin have the touch of the Eighteenth Century, but his work breathes the spirit of a new period. He stood, with his contemporaries

Courtesy of Scripta Mathematica

CARL FRIEDRICH GAUSS (1777–1855)

Kant, Goethe, Beethoven, and Hegel, on the side lines of a great political struggle raging in other countries, but expressed in his own field the new ideas of his age in a most powerful way.

Gauss' diaries show that already in his seventeenth year he began to make startling discoveries. In 1795, for instance, he discovered independently of Euler the law of quadratic reciprocity in number theory. Some of his early discoveries were published in his Helmstädt dissertation of 1799 and in the impressive "Disquisitiones arithmeticae" of 1801. The dissertation gave the first rigorous proof of the so-called fundamental theorem of algebra, the theorem that every algebraic equation with real coefficients has at least one root and hence has n roots. The theorem itself goes back to Albert Girard, the editor of Stevin's works ("Invention nouvelle en algèbre", 1629), D'Alembert had tried to give a proof in 1746. Gauss loved this theorem and later gave two more demonstrations, returning in 1846 to his first proof. The third demonstration (1816) used complex integrals and showed Gauss' early mastery of the theory of complex numbers.

The "Disquisitiones arithmeticae" collected all the masterful work in number theory of Gauss' predecessors and enriched it to such an extent that the beginning of modern number theory is sometimes dated from the publication of this book. Its core is the theory of quadratic congruences, forms, and residues; it culminates in the law of quadratic residues, that "theorema aureum" of which Gauss gave the first complete proof. Gauss was as fascinated by this theorem as by the fundamental theorem of algebra and later published five

more demonstrations; one more was found after his death among his papers. The "Disquisitiones" also contain Gauss' studies on the division of the circle, in other words, on the roots of the equation $x^n = 1$. They led up to the remarkable theorem that the sides of the regular polygon of 17 sides (more general, of n sides, $n = 2^p + 1, p = 2^k$, n prime, $k = 0, 1, 2, 3 \cdots$) can be constructed with compass and ruler alone, a striking extension of the Greek type of geometry.

Gauss' interest in astronomy was aroused when, on the first day of the new century, on Jan. 1, 1801, Piazzi in Palermo discovered the first planetoid, which was given the name of Ceres. Since only a few observations of the new planetoid could be made the problem arose to compute the orbit of a planet from a smaller number of observations. Gauss solved the problem completely; it leads to an equation of degree eight. When in 1802, Pallas, another planetoid, was discovered, Gauss began to interest himself in the secular perturbations of planets. This led to the "Theoria motus corporum coelestium" (1809), to his paper on the attraction of general ellipsoids (1813), to his work on mechanical quadrature (1814), and to his study of secular perturbations (1818). To this period belongs also Gauss' paper on the hypergeometric series (1812), which allows a discussion of a large number of functions from a single point of view. It is the first systematic investigation of the convergence of a series.

After 1820 Gauss began to be actively interested in geodesy. Here he combined extensive applied work in triangulation with his theoretical research. One of the

results was his exposition of the method of least squares (1821, 1823), already the subject of investigation by Legendre (1806) and Laplace. Perhaps the most important contribution of this period in Gauss' life was the surface theory of the "Disquisitiones generales circa superficies curvas" (1827), which approached its subject in a way strikingly different from that of Monge. Here again practical considerations, now in the field of higher geodesy, were intimately connected with subtle theoretical analysis. In this publication appeared the so-called intrinsic geometry of a surface, in which curvilinear coordinates are used to express the linear element ds in a quadratic differential form $ds^2 = Edu^2 + Fdu\, dv + Gdv^2$. There was also a climax, the "theorema egregium", which states that the total curvature of the surface depends only on E, F, and G and their derivatives, and thus is a bending invariant. But Gauss did not neglect his first love, the "queen of mathematics," even in this period of concentrated activity on problems of geodesy, for in 1825 and 1831 appeared his work on biquadratic residues. It was a continuation of his theory of quadratic residues in the "Disquisitiones arithmeticae," but a continuation with the aid of a new method, the theory of complex numbers. The treatise of 1831 did not only give an algebra of complex numbers, but also an arithmetic. A new prime number theory appeared, in which 3 remains prime but $5 = (1 + 2i)(1 - 2i)$ is no longe a prime. This new complex number theory clarified many dark points in arithmetic so that the quadratic law of reciprocity became simpler than in real numbers. In this paper

Gauss dispelled forever the mystery which still surrounded complex numbers by his representation of them by points in a plane.[1]

A statue in Göttingen represents Gauss and his younger colleague Wilhelm Weber in the process of inventing the electric telegraph. This happened in 1833-34 at a time when Gauss' attention began to be drawn toward physics. In this period he did much experimental work on terrestrial magnetism. But he also found time for a theoretical contribution of the first importance—his "Allgemeine Lehrsätze" on the theory of forces acting inversely proportional to the square of the distance (1839, 1840). This was the beginning of potential theory as a separate branch of mathematics (Green's paper in 1828 was practically unknown at that time) and it led to certain minimal principles concerning space integrals, in which we recognize "Dirichlet's" principle. For Gauss the existence of a minimum was evident; this later became a much debated question which was finally solved by Hilbert.

Gauss remained active until his death in 1855. In his later years he concentrated more and more on applied mathematics. His publications, however, do not give an adequate picture of his full greatness. The appearance of his diaries and of some of his letters has shown that he kept some of his most penetrating thoughts to himself. We now know that Gauss, as early as 1800, had discovered elliptic functions and around 1816 was in possession of non-euclidean geometry. He never published anything on these subjects; indeed, only in some letters to friends did he disclose his critical position toward attempts to prove Euclid's parallel axiom. Gauss seems to have been unwilling to venture publicly into any controversial subject. In letters he wrote about the wasps who would then fly around his ears and of the "shouts of the Boeotians" that would be heard if his secrets were not kept. For himself Gauss doubted the validity of the accepted Kantian doctrine that space conception is euclidean a priori; for him the real geometry of space was a physical fact to be discovered by experiment.

[1]Comp. E. T. Bell, *Gauss and the Early Development of Algebraic Numbers*, National Math. Magazine 18 (1944) pp. 188, 219.

4. In his history of mathematics of the Nineteenth Century Felix Klein has invited comparison between Gauss and the twenty-five year older French mathematician Adrien Marie Legendre. It is perhaps not entirely fair to compare Gauss with any mathematician except the very greatest; but this particular comparison shows how Gauss' ideas were "in the air," since Legendre in his own independent way worked on most subjects which occupied Gauss. Legendre taught from 1775 to 1780 at the military school in Paris and later filled different government positions such as professor at the Ecole Normale, examiner at the Ecole Polytechnique, and geodetic surveyor.

Like Gauss he did fundamental work on number theory ("Essai sur les nombres," 1798 "Theorie des nombres," 1830) where he gave a formulation of the law of quadratic reciprocity. He also did important work on geodesy and on theoretical astronomy, was as assiduous a computer of tables as Gauss, formulated in

Courtesy of Scripta Mathematica

ADRIEN MARIE LEGENDRE (1752–1833)

1806 the method of least squares, and studied the attraction of ellipsoids—even those which are not surfaces of revolution. Here he introduced the "Legendre" functions. He also shared Gauss' interest in elliptic and Eulerian integrals as well as in the foundations and methods of Euclidean geometry.

Although Gauss penetrated deeper into the nature of all these different fields of mathematics, Legendre produced works of outstanding importance. His comprehensive textbooks were for a long time authoritative, especially his "Exercises du calcul intégral" (3 vols, 1811-19) and his "Traité des fonctions elliptiques et des intégrales eulériennes" (1827-32), still a standard work. In his "Elements de géométrie" (1794) he broke with the Platonic ideals of Euclid and presented a textbook of elementary geometry based on the requirements of modern education. This book passed through many editions and was translated into several languages; it has had a lasting influence.

5. The beginnings of the new period in the history of French mathematics may perhaps be dated from the establishment of military schools and academies, which took place during the later part of the Eighteenth Century. These schools, of which some were also founded outside of France (Turin, Woolwich), paid considerable attention to the teaching of mathematics as a part of the training of military engineers. Lagrange started his career at the Turin school of artillery; Legendre and Laplace taught at the military school in Paris, Monge at Mézières. Carnot was a captain of engineers. Napoleon's interest in mathematics dates back to his student days

at the military academies of Brienne and Paris. During the invasion of France by the Royalist armies the need of a more centralized instruction in military engineering became apparent. This led to the foundation of the Ecole Polytechnique of Paris (1794), a school which soon developed into a leading institution for the study of general engineering and eventually became the model of all engineering and military schools of the early Nineteenth Century, including West Point in the United States. Instruction in theoretical and applied mathematics was an integral part of the curriculum. Emphasis was laid upon research as well as upon teaching. The best scientists of France were induced to lend their support to the school; many great French mathematicians were students, professors, or examiners at the Ecole Polytechnique.[1]

The instruction at this institution, as well as at other technical schools, required a new type of textbook. The learned treatises for the initiated which were so typical of Euler's period had to be supplemented by college handbooks. Some of the best textbooks of the early Nineteenth Century were prepared for the instruction at the Ecole Polytechnique or similar institutions. Their influence can be traced in our present day texts. A good example of such a handbook is the "Traité du calcul différentiel et du calcul intégral" (2 vols., 1797) written by Sylvestre François Lacroix, from which whole generations have learned their calculus. We have already mentioned Legendre's books. A further example is Monge's textbook of descriptive geometry, which still is followed by many present day books on this subject.

[1]Comp. C. G. J. Jacobi, *Werke* 7, p. 355 (lecture held in 1835).

6. Gaspard Monge, the director of the Ecole Polytechnique, was the scientific leader of the group of mathematicians which were connected with this institute. He had started his career as instructor at the military academy of Mézières (1768-1789), where his lectures on fortification gave him an opportunity to develop descriptive geometry as a special branch of geometry. He published his lectures in the "Géométrie descriptive" (1795-1799). At Mézières he also began to apply the calculus to space curves and surfaces, and his papers on this subject were later published in the "Application de l'analyse à la géométrie" (1809), the first book on differential geometry, though not yet in the form which is customary at present. Monge was one of the first modern mathematicians whom we recognize as a specialist: a geometer—even his treatment of partial differential equations has a distinctly geometrical touch.

Through Monge's influence geometry began to flourish at the Ecole Polytechnique. In Monge's descriptive geometry lay the nucleus of projective geometry, and his mastery of algebraic and analytical methods in their application to curves and surfaces contributed greatly to analytical and differential geometry. Jean Hachette and Jean Baptiste Biot developed the analytical geometry of conics and quadrics; in Biot's "Essai de géométrie analytique" (1802) we begin at last to recognize our present textbooks of analytic geometry. Monge's pupil, Charles Dupin, as a young naval engineer in Napoleonic days, applied his teacher's methods to the theory of surfaces, where he found the asymptotic and conjugate lines. Dupin became a professor of

MONGE.

geometry in Paris and gained during his long life prominence as a politician and industrial promoter as well. The "indicatrix of Dupin" and the "cyclides of Dupin" remind us of the early interests of this man, whose "Développements de géométrie" (1813) and "Applications de géométrie" (1825) contain a great number of interesting ideas.

The most original Monge pupil was Victor Poncelet. He had an opportunity to reflect on his teacher's methods during 1813 when he lived an isolated existence as a war prisoner in Russia after the defeat of Napoleon's "grande armée." Poncelet was attracted by the purely synthetic side of Monge's geometry and thus was led to a mode of thinking already suggested two centuries before by Desargues. Poncelet became the founder of projective geometry.

Poncelet's "Traité des propriétés projectives des figures" appeared in 1822. This heavy volume contains all the essential concepts underlying the new form of geometry, such as cross ratio, perspectivity, projectivity, involution, and even the circular points at infinity. Poncelet knew that the foci of a conic can be considered as the intersections of the tangents at the conic through these circular points. The "Traité" also contains the theory of the polygons inscribed in one conic and circumscribed to another one (the so-called closure problem of Poncelet). Although this book was the first full treatise on projective geometry, during the next decades this geometry reached that degree of perfection which made it a classical example of a well-integrated mathematical structure.

6. Monge, although a man of strict democratic principles, stayed loyal to Napoleon, in whom he saw the executor of the ideals of the Revolution. In 1815, when the Bourbons returned, Monge lost his position and soon afterwards died. The Ecole Polytechnique, however, continued to flourish in Monge's spirit. The very nature of the instruction made it difficult to separate pure and applied mathematics. Mechanics received full attention, and mathematical physics began at last to emancipate itself from the "catoptrics" and the "dioptrics" of the Ancients. Etienne Malus discovered the polarization of light (1810) and Augustin Fresnel reestablished Huygens' undulatory theory of light (1821). André Marie Ampère, who had done distinguished work on partial differential equations, became after 1820 the great pioneer in electro-magnetism. These investigators brought many direct and indirect benefits to mathematics: one example is Dupin's improvement of Malus' geometry of light rays, which helped to modernize geometrical optics and also contributed to the geometry of line congruences.

Lagrange's "Mécanique analytique" was faithfully studied and its methods tested and applied. Statics appealed to Monge and his pupils because of its geometrical possibilities, and several textbooks on statics appeared in the course of the years, including one by Monge himself (1788, many editions). The geometrical element in statics was brought out in full by Louis Poinsot, for many years a member of the French superior board of public instruction. His "Elements de statique" (1804) and "Theorie nouvelle de la rotation des corps" (1834) added to the conception of force that of torque ("couple"), represented Euler's theory of moments of inertia by means of the ellipsoid of inertia, and analyzed the motion of this ellipsoid when the rigid body moves in space or turns about a fixed point. Poncelet and Coriolis gave a geometrical touch to Lagrange's analytical mechanics; both men, as well as Poinsot, stressed the application of mechanics to the theory of simple machines. The "acceleration of Coriolis", which appears when a body moves in an accelerated system, is an example of such a geometrical interpretation of Lagrange's results (1835).

The most outstanding mathematicians connected with the early years of the Ecole Polytechnique were—apart from Lagrange and Monge—Siméon Poisson, Joseph Fourier, and Augustin Cauchy. All three were deeply interested in the application of mathematics to mechanics and physics, all three were led by this interest to discoveries in "pure" mathematics. Poisson's productivity is indicated by the frequency in which his name occurs in our textbooks: Poisson's brackets in differential equations, Poisson's constant in elasticity, Poisson's integral, and Poisson's equation in potential theory. This "Poisson equation," $\Delta V = 4\pi\rho$, was the result of Poisson's discovery (1812) that Laplace's equation, $\Delta V = 0$, only holds outside of the masses; its exact proof for masses of variable density was not given until Gauss gave it in his "Allgemeine Lehrsätze" (1839–40). Poisson's "Traité de mécanique" (1811) was written in the spirit of Lagrange and Laplace but contained many innovations, such as the explicit use of

impulse coordinates, $p_i = \partial T/\partial \dot{q}_i$, which later inspired the work of Hamilton and Jacobi. His book of 1837 contains "Poisson's law" in probability (see p. 145).

Fourier is primarily remembered as the author of the "Théorie analytique de la chaleur" (1822). This is the mathematical theory of heat conduction and, therefore, is essentially the study of the equation $\Delta U = k\partial u/\partial t$. By virtue of the generality of its method this book became the source of all modern methods in mathematical physics involving the integration of partial differential equations under given boundary conditions. This method is the use of trigonometric series, which had been the cause of discussion between Euler, D'Alembert, and Daniel Bernoulli. Fourier made the situation perfectly clear. He established the fact that an "arbitrary" function (a function capable of being represented by an arc of a continuous curve or by a succession of such arcs) could be represented by a trigonometric series of the form, $\sum_{n=0}^{\infty} (A_n \cos nax + B_n \sin nax)$. Despite Euler's and Bernoulli's observations the idea was so new and startling at the time of Fourier's investigations that it is said that when he stated his ideas in 1807 for the first time, he met with the vigorous opposition of no one other than Lagrange himself.

The "Fourier series" now became a well-established instrument of operation in the theory of partial differential equations with given boundary conditions. It also received attention on its own merits. Its manipulation by Fourier fully opened the question of what to understand by a "function." This was one of the reasons why Nineteenth Century mathematicians found it necessary to look closer into questions concerning rigor of mathematical proof and the foundation of mathematical conceptions in general.[1] This task, in the specific case of Fourier series, was undertaken by Dirichlet and Riemann.

7. Cauchy's many contributions to the theory of light and to mechanics have been obscured by the success of his work in analysis, but we must not forget that with Navier he belongs to the founders of the mathematical theory of elasticity. His main glory is the theory of functions of a complex variable and his insistence on rigor in analysis. Functions of a complex variable had already been constructed before, notably by D'Alembert, who in a paper on the resistance of fluids of 1752 even had obtained what we now call the Cauchy-Riemann equations. In Cauchy's hands complex function theory emancipated from a tool useful in hydrodynamics and aerodynamics into a new and independent field of mathematical research. Cauchy's investigations on this subject appeared in constant succession after 1814. One of the most important of his papers is the "Mémoire sur les intégrales définies, prises entre des limites imaginaires" (1825). In this paper appeared Cauchy's integral theorem with residues. The theorem that every regular function $f(z)$ can be expanded around each point $z = z_0$ in a series convergent in a circle passing through the singular point nearest to $z = z_0$ was published in 1831, the same year in which Gauss published his arithmetical theory of complex numbers. Laurent's

[1] P. E. B. Jourdain, Note on Fourier's Influence on the Conceptions of Mathematics, Proc. Intern. Congress of Mathem. (Cambridge, 1912) II, pp. 526-527.

extension of Cauchy's theorem on series was published in 1843—when it was also in the possession of Weierstrass. These facts show that Cauchy's theory did not have to cope with professional resistance; the theory of complex functions was fully accepted from its very beginning.

Cauchy, along with his contemporaries Gauss, Abel, and Bolzano, belongs to the pioneers of the new insistence on rigor in mathematics. The Eighteenth Century had been essentially a period of experimentation, in which results came pouring in with luxurious abundance. The mathematicians of this age had not bothered too much about the foundation of their work—"allez en avant, et la foi vous viendra," D'Alembert is supposed to have said. When they worried about rigor, as Euler and Lagrange occasionally did, their arguments were not always convincing. The time had now arrived for a close concentration on the meaning of the results. What was a "function" of a real variable, which showed such different behavior in the case of a Fourier series and in the case of a power series? What was its relation to the entirely different "function" of a complex variable? These questions brought all the unsolved problems about the foundation of the calculus and the existence of the potentially and the actually infinite again into the foreground of mathematical thinking.[1] What Eudoxos had done in the period after the fall of Athenian democracy, Cauchy and his meticulous contemporaries began to accomplish in the period of

[1]P. E. B. Jourdain, *The Origin of Cauchy's Conception of a Definite Integral and of the Continuity of a Function*, Isis 1 (1913) pp. 661-703 (see also Bibl. Math. 6 (1905) pp. 190-207).

expanding industrialism. This difference of social setting produced different results, and where Eudoxos' success had the tendency to stifle productivity, the success of the modern reformers stimulated mathematical productivity to a high degree. Cauchy and Gauss were followed by Weierstrass and Cantor.

Cauchy gave the foundation of the calculus as we now generally accept it in our textbooks. It can be found in his "Cours d'analyse" (1821) and his "Résume des leçons données à l'école royale polytechnique" I (1823). Cauchy used D'Alembert's limit concept to define the derivative of a function, and establish it on a firmer foundation than his predecessors had been able to do.

Starting with the definition of a *limit*, Cauchy gave examples such as the limit of $\sin \alpha/\alpha$ for $\alpha = 0$. Then he defined a "variable infiniment petite" as a variable number which has zero for its limit; and then postulated that Δy and Δx "seront des quantités infiniment petites". He then wrote $\frac{\Delta y}{\Delta x} = \frac{f(x+i) - f(x)}{i}$ and called the limit for $i \to 0$, the "fonction dérivée, y' ou $f'(x)$." He placed $i = \alpha h$, α an "infiniment petite", and h a "quantité finie":

$$\frac{f(x+\alpha h) - f(x)}{\alpha} = \frac{f(x+i) - f(x)}{i} h$$

h was called the "différentielle de la fonction $y = f(x)$." Furthermore, $dy = df(x) = h f'(x)$; $dx = h$.[1]

Cauchy used both Lagrange's notation and many of

[1]Resumé I (1823) Calcul différentiel, pp. 13-27. A strict analysis of this procedure in M. Pasch, *Mathematik am Ursprung* (Leipzig, 1927) pp. 47-73.

his contributions to real function theory without making any concession to Lagrange's "algebraic" foundation. The mean value theorem and the remainder of the Taylor series were accepted as Lagrange had derived them, but series were now discussed with due attention to their convergence. Several convergence proofs in the theory of infinite series are named after Cauchy. There are definite steps in his books toward that "arithmetization" of analysis which later became the core of Weierstrass' investigations. Cauchy also gave the first existence proof for the solution of a differential equation and of a system of such equations (1836). In this way Cauchy offered at last a beginning of an answer to that series of problems and paradoxes which had haunted mathematics since the days of Zeno, and he did this not denying or ignoring them, but by creating a mathematical technique in which it was possible to do them justice.

Cauchy, like his contemporary Balzac with whom he shared a capacity for an infinite amount of work, was a legitimist and a royalist. Both men had such a deep understanding of values that despite their reactionary ideals much of their work has retained its fundamental place. Cauchy abandoned his chair at the Ecole Polytechnique after the Revolution of 1830 and spent some years at Turin and Prague; he returned to Paris in 1838. After 1848 he was allowed to stay and teach without having to take the oath of allegiance to the new government. His productivity was so enormous that the Paris Academy had to restrict the size of all papers sent to the "Comptes Rendus" in order to cope with Cauchy's output. It is told that he disturbed Laplace

so greatly when he read his first paper on the convergence of series to the Paris Academy that the great man went back home to test the series in his "Mécanique Céleste". He found, it seems, that no great errors had been committed.

8. This Paris milieu with its intense mathematical activity produced, around 1830, a genius of the first order, who, comet-like, disappeared as suddenly as he had appeared. Evariste Galois, the son of a small-town mayor near Paris, was twice refused admission to the Ecole Polytechnique and succeeded at last in entering the Ecole Normale only to be dismissed. He tried to make a living by tutoring mathematics, maintaining at the same time an uneasy balance between his ardent love for science and for democracy. Galois participated as a republican in the Revolution of 1830, spent some months in prison, and was soon afterwards killed in a duel at the age of twenty-one. Two of the papers he sent for publication got lost on the editor's desk; some others were published long after his death. On the eve of the duel, he wrote to a friend a summary of his discoveries in the theory of equations. This pathetic document, in which he asked his friend to submit his discoveries to the leading mathematicians, ended with the words:

> "You will publicly ask Jacobi or Gauss to give their opinion not on the truth, but on the importance of the theorems. After this there will be, I hope, some people who will find it to their advantage to decipher all this mess."

This mess ("ce gâchis") contained no less than the

EVARISTE GALOIS (1811–1832)
From a rare sketch showing him as he appeared shortly before his fatal duel at the age of twenty-one

theory of groups, the key to modern algebra and to modern geometry. The ideas had already been anticipated to a certain extent by Lagrange and the Italian Ruffini, but Galois had the conception of a complete theory of groups. He expressed the fundamental properties of the transformation group belonging to the roots of an algebraic equation and showed that the field of rationality of these roots was determined by the group. Galois pointed out the central position taken by invariant sub-groups. Ancient problems such as the trisection of the angle, the duplication of the cube, the solution of the cubic and biquadratic equation, as well as the solution of an algebraic equation of any degree found their natural place in the theory of Galois. Galois' letter, as far as we know, was never submitted to Gauss or Jacobi. It never reached a mathematical public until Liouville published most of Galois' papers in his "Journal de mathématiques" of 1846, at which period Cauchy had already begun to publish on group theory (1844-46). It was only then that some mathematicians began to be interested in Galois' theories. Full understanding of Galois' importance came only through Camille Jordan's "Traité des substitutions" (1870) and the subsequent publications by Klein and Lie. Now Galois' unifying principle has been recognized as one of the outstanding achievements of Nineteenth Century mathematics.[1]

Galois also had ideas on the integrals of algebraic functions of one variable, on what we now call Abelian integrals. This brings his manner of thinking closer

[1]See G. A. Miller, *History of the Theory of Groups to 1900.* Coll. Works I (1935) pp. 427-467.

to that of Riemann. We may speculate on the possibility that if Galois had lived, modern mathematics might have received its deepest inspiration from Paris and the school of Lagrange rather than from Göttingen and the school of Gauss.

9. Another young genius appeared in the twenties, Niels Henrik Abel, the son of a Norwegian country minister. Abel's short life was almost as tragic as that of Galois. As a student in Christiania he thought for a while that he had solved the equation of degree five, but he corrected himself in a pamphlet published in 1824. This was the famous paper in which Abel proved the impossibility of solving the general quintic equation by means of radicals—a problem which had puzzled the mathematicians from the time of Bombelli and Viète (a proof of 1799 by the Italian Paolo Ruffini was considered by Poisson and other mathematicians as too vague). Abel now obtained a stipend which enabled him to travel to Berlin, Italy, and France. Tortured by poverty and consumption, shy and retiring, the young mathematician established few personal contacts and died (1829) soon after his return to his native land.

In this period of travel Abel wrote several papers which contain his work on the convergence of series, on "Abelian" integrals, and on elliptic functions. Abel's theorems in the theory of infinite series show that he was able to establish this theory on a reliable foundation. "Can you imagine anything more horrible than to claim that $0 = 1^n - 2^n + 3^n - 4^n + \text{etc.}$, n being a positive integer?" he wrote to a friend, and he continued:

"There is in mathematics hardly a single infinite series of which the sum is determined in a rigorous way" (letter to Holmboe, 1826).

Abel's investigations on elliptic functions were conducted in a short but exciting competition with Jacobi. Gauss in his private notes had already found that the inversion of elliptic integrals leads to single-valued, doubly periodic functions, but he never published his ideas. Legendre, who had spent so much effort on elliptic integrals, had missed this point entirely and was deeply impressed when, as an old man, he read Abel's discoveries. Abel had the good luck to find a new periodical eager to print his papers; the first volume of the "Journal für die reine und angewandte Mathematik" edited by Crelle contained no less than five of Abel's papers. In the second volume (1827) appeared the first part of his "Recherches sur les fonctions elliptiques," with which the theory of doubly periodic functions begins.

We speak of Abel's integral equation and of Abel's theorem on the sum of integrals of algebraic functions which leads to Abelian functions. Commutative groups are called Abelian groups, which indicates how close Galois' ideas were related to those of Abel.

10. In 1829, the year that Abel died, Carl Gustav Jacob Jacobi published his "Fundamenta nova theoriae functionum ellipticarum." The author was a young professor at the University of Königsberg. He was the son of a Berlin banker and a member of a distinguished family; his brother Moritz in St. Petersburg was one of

the earliest Russian scientists who experimented in electricity. After studying in Berlin, Jacobi taught at Königsberg from 1826 to 1843. He then spent some time in Italy trying to regain his health and ended his career as a professor at the University of Berlin, dying in 1851 at the age of forty-six. He was a witty and liberal thinker, an inspiring teacher, and a scientist whose enormous energy and clarity of thought left few branches of mathematics untouched.

Jacobi based his theory of elliptic functions on four functions defined by infinite series and called theta functions. The doubly periodic functions *sn u*, *cn u* and *dn u* are quotients of theta functions; they satisfy certain identities and addition theorems very much like the sine and cosine functions of ordinary trigonometry. The addition theorems of elliptic functions can also be considered as special applications of Abel's theorem on the sum of integrals of algebraic functions. The question now arose whether hyper-elliptic integrals could be inverted in the way elliptic integrals had been inverted to yield elliptic functions. The solution was found by Jacobi in 1832 when he published his result that the inversion could be performed with functions of more than one variable. Thus the theory of Abelian functions of p variables was born, which became an important branch of Nineteenth Century mathematics.

Sylvester has given the name "Jacobian" to the functional determinant in order to pay respect to Jacobi's work on algebra and elimination theory. The best known of Jacobi's papers on this subject is his "De formatione et proprietatibus determinantium"

(1841), which made the theory of determinants the common good of the mathematicians. The idea of the determinant was much older—it goes back essentially to Leibniz (1693), the Swiss mathematician Gabriel Cramer (1750), and Lagrange (1773); the name is due to Cauchy (1812). Y. Mikami has pointed out that the Japanese mathematician, Seki Kōwa, had the idea of a determinant sometime before 1683.[1]

The best approach to Jacobi is perhaps through his beautiful lectures on dynamics ("Vorlesungen über Dynamik"), published in 1866 after lecture notes from 1842-43. They are written in the tradition of the French school of Lagrange and Poisson but have a wealth of new ideas. Here we find Jacobi's investigations on partial differential equations of the first order and their application to the differential equations of dynamics. An interesting chapter of the "Vorlesungen über Dynamik" is the determination of the geodesics on an ellipsoid; the problem leads to a relation between two Abelian integrals.

11. Jacobi's lectures on dynamics lead us to another mathematician whose name is often linked with that of Jacobi, William Rowan Hamilton (not to be confused with his contemporary, William Hamilton, the Edinburgh philosopher). He lived his whole life in Dublin, where he was born of Irish parents. He entered Trinity College, became in 1827 at the age of twenty-one Royal Astronomer of Ireland and held this position until his death in 1865. As a boy he learned continental mathe-

[1] Y. Mikami, *On the Japanese Theory of Determinants*, Isis 2 (1914) pp. 9-36.

matics—still a novelty in the United Kingdom—by studying Clairaut and Laplace and showed his mastery of the novel methods in his extremely original work on optics and dynamics. His theory of optical rays (1824) was more than merely a differential geometry of line congruences; it was also a theory of optical instruments and allowed Hamilton to predict the conical refractions in biaxial crystals. In this paper appeared his "characteristic function," which became the guiding idea of the "General Method in Dynamics," published in 1834-35. Hamilton's idea was to derive both optics and dynamics from one general principle. Euler, in his defense of Maupertuis, had already shown how the stationary value of the action integral could serve this purpose. Hamilton, in accordance with this suggestion, made optics and dynamics two aspects of the calculus of variations. He asked for the stationary value of a certain integral and considered it as a function of its limits. This was the "characteristic" or "principal" function, which satisfied two partial differential equations. One of these partial differential equations, which is usually written

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial q}, q\right) = 0$$

was specially selected by Jacobi for his lectures on dynamics and is now known as the Hamilton-Jacobi equation. This has obscured the importance of Hamilton's characteristic function, which had the central place in his theory as a means of unifying mechanics and mathematical physics. It was rediscovered by Bruns in 1895 in the case of geometrical optics and as "eikonal" has shown its use in the theory of optical instruments.

The part of Hamilton's work on dynamics which has passed into the general body of mathematics is, in the first place, the "canonical" form, $\dot{q} = \partial H/\partial p$, $\dot{p} = -\partial H/\partial q$, in which he wrote the equations of dynamics. Canonical form and Hamilton-Jacobi differential equation have enabled Lie to establish the relation between dynamics and contact transformations. Another since equally accepted idea of Hamilton was the derivation of the laws of physics and mechanics from the variation of an integral. Modern relativity, as well as quantum mechanics, has based itself on "Hamiltonian" functions as its underlying principle.

The year 1843 was a turning point in Hamilton's life. In that year he found the quaternions, to the study of which he devoted the later part of his life. We shall discuss this discovery later.

12. Peter Lejeune Dirichlet was closely associated with Gauss and Jacobi, as well as with the French mathematicians. He lived from 1822-27 as a private tutor and met Fourier, whose book he studied; he also became familiar with Gauss' "Disquisitiones arithmeticae." He later taught at the University of Breslau and in 1855 succeeded Gauss at Göttingen. His personal acquaintance with French as well as German mathematics and mathematicians made him the appropriate man to serve as the interpreter of Gauss and to subject Fourier series to a penetrating analysis. Dirichlet's beautiful "Vorlesungen über Zahlentheorie" (publ. 1863) still form one of the best introductions into Gauss' investigations in number theory. They also contain many new results.

In a paper of 1840 Dirichlet showed how to apply the full power of the theory of analytical functions to problems in number theory; it was in these investigations that he introduced the "Dirichlet" series. He also extended the notion of quadratic irrationalities to that of general algebraic domains of rationality.

Dirichlet first gave a rigorous convergence proof of Fourier series, and in this way contributed to a correct understanding of the nature of a function. He also introduced into the calculus of variations the so-called Dirichlet principle, which postulated the existence of a function v which minimizes the integral $\int[v_x^2 + v_y^2 + v_z^2]d\tau$ under given boundary conditions. It was a modification of a principle which Gauss had introduced in his potential theory of 1839-40, and it later served Riemann as a powerful tool in solving problems in potential theory. We have already mentioned that Hilbert eventually established rigorously the validity of this principle (p. 208).

13. With Bernhard Riemann, Dirichlet's successor at Göttingen, we reach the man who more than any other has influenced the course of modern mathematics. Riemann was the son of a country minister and studied at the University of Göttingen, where in 1851 he obtained the doctor's degree. In 1854 he became Privatdozent, in 1859 a professor at the same university. He was sickly, like Abel, and spent his last days in Italy, where he died in 1866 at forty years of age. In his short life he published only a relatively small number of papers but each of them was—and is—important, and several have opened entirely new and productive fields.

GEORG FRIEDRICH BERNHARD RIEMANN (1826–1866)

In 1851 appeared Riemann's doctoral thesis on the theory of complex functions $u + iv = f(x + iy)$. Like D'Alembert and Cauchy, Riemann was influenced by hydrodynamical considerations. He mapped the (xy)-plane conformally on the (uv)-plane and established the existence of a function capable of transforming any simply connected region in one plane into any simply connected region in the other plane. This led to the conception of the Riemann surface, which introduced topological considerations into analysis. At that time topology was still an almost untouched subject on which J. B. Listing had published a paper in the "Göttingen Studien" of 1847. Riemann showed its central importance for the theory of complex functions. This thesis also clarified Riemann's definition of a complex function: its real and imaginary part have to satisfy the "Cauchy-Riemann" equations, $u_x = v_y$, $u_y = -v_x$, in a given region, and furthermore have to satisfy certain conditions as to boundary and singularities.

Riemann applied his ideas to hypergeometric and Abelian functions (1857), using freely Dirichlet's principle (as he called it). Among his results is the discovery of the genus of a Riemann surface as a topological invariant and as a means of classifying Abelian functions. A posthumously published paper applies his ideas to minimal surfaces (1867). To this branch of Riemann's activity also belong his investigations on elliptical modular-functions and Θ-series in p independent variables, as well as those on linear differential equations with algebraic coefficients.

Riemann became a Privatdozent in 1850 by submitting no less than two fundamental papers, one on trigonometric series and the foundations of analysis, the other one on the foundations of geometry. The first of these papers analyzed Dirichlet's conditions for the expansion of a function in a Fourier series. One of these conditions was that the function be "integrable." But what does this mean? Cauchy and Dirichlet had already given certain answers; Riemann replaced them by his own, more comprehensive one. He gave that definition which we now know as the "Riemann integral," and which was replaced only in the Twentieth Century by the Lebesgue integral. Riemann showed how functions, defined by Fourier series, may show such properties as the possession of an infinite number of maxima or minima, which older mathematicians would not have accepted in their definition of a function. The concept of a function began seriously to emancipate from the "curva quaecunque libero manus ductu descripta"[1] of Euler. In his lectures Riemann gave an example of a continuous function without derivatives; an example of such a function which Weierstrass had given was published in 1875. Mathematicians refused to take such functions very seriously and called them "pathological" functions; modern analysis has shown how natural such functions are and how Riemann here again had penetrated into a fundamental field of mathematics.

The other paper of 1854 deals with the hypotheses on which geometry is based. Space was introduced as a topological manifold of an arbitrary number of dimensions; a metric was defined in such a manifold by means of a quadratic differential form. Where Riemann,

[1] "Some curve described by freely leading the hand" (Inst. Calc. integr. III §301).

in his analysis, had defined a complex function by its local behavior, in this paper he defined the character of space in the same way. Riemann's unifying principle not only enabled him to classify all existing forms of geometry, including the still very obscure non-euclidean geometry, but also allowed the creation of any number of new types of space, many of which have since found a useful place in geometry and mathematical physics. Riemann published this paper without any analytical technique, which made his ideas difficult to follow. Later some of the formulas appeared in a prize essay on the distribution of heat on a solid, which Riemann submitted to the Paris Academy (1861). Here we find a sketch of the transformation theory of quadratic forms.

The last paper of Riemann which must be mentioned is his discussion of the number $F(x)$ of primes less than a given number x (1859). It was an application of complex number theory to the distribution of primes and analyzed Gauss' suggestion that $F(x)$ approximates the logarithmic integral $\int_2^x (\log t)^{-1}\, dt$. This paper is celebrated because it contains the so-called Riemann hypothesis that Euler's Zeta-function $\zeta(s)$—the notation is Riemann's—if considered for complex $s = x + iy$, has all non-real zeros on the line $x = \frac{1}{2}$. This hypothesis has never been proved, nor has it been disproved.[1]

14. Riemann's concept of the function of a complex variable has often been compared to that of Weierstrass. Karl Weierstrass was for many years a

[1] R. Courant, *Bernhard Riemann und die Mathematik der letzten hundert Jahre*, Naturwissch. 14 (1926) pp. 813-818.

teacher at Prussian Gymnasia (Latin high schools) and in 1856 became professor of mathematics at the University of Berlin, where he taught for thirty years. His lectures, always meticulously prepared, enjoyed increasing fame; it is mainly through these lectures that Weierstrass' ideas have become the common property of mathematicians.

In his Gymnasial period Weierstrass wrote several papers on hyperelliptic integrals, Abelian functions, and algebraic differential equations. His best known contribution is his foundation of the theory of complex functions on the power series. This was in a certain sense a return to Lagrange, with the difference that Weierstrass worked in the complex plane and with perfect rigor. The values of the power series inside its circle of convergence represent the "function element," which is then extended, if possible, by so-called analytic continuation. Weierstrass especially studied entire functions and functions defined by infinite products. His elliptic function $\wp(u)$ has become as established as the older $sn\ u$, $cn\ u$, $dn\ u$ of Jacobi.

Weierstrass' fame has been based on his extremely careful reasoning, on "Weierstrassian rigor," which is not only apparent in his function theory but also in his calculus of variations. He clarified the notions of minimum, of function, and of derivative, and with this eliminated the remaining vagueness of expression in the fundamental concepts of the calculus. He was the mathematical conscience par excellence, methodological and logical. Another example of his meticulous reasoning is his discovery of uniform convergence. With Weierstrass began that reduction of the principles of

KARL WEIERSTRASS (1815–1897)

analysis to the simplest arithmetical concepts which we call the arithmetization of mathematics.

"It is essentially a merit of the scientific activity of Weierstrass that there exists at present in analysis full agreement and certainty concerning the course of such types of reasoning which are based on the concept of irrational number and of limit in general. We owe it to him that there is unanimity on all results in the most complicated questions concerning the theory of differential and integral equations, despite the most daring and diversified combinations with application of super-, juxta-, and transposition of limits."[1]

15. This arithmetization was typical of the so-called School of Berlin and especially of Leopold Kronecker. To this school belonged such eminent mathematicians, proficient in algebra and the theory of algebraic numbers, as Kronecker, Kummer, and Frobenius. With these men we may associate Dedekind and Cantor. Ernst Kummer was called to Berlin in 1855 as successor to Dirichlet; he taught there until 1883, when he voluntarily stopped doing mathematical work because he felt a coming decline in productivity. Kummer further developed the differential geometry of congruences which Hamilton had outlined and in the course of his study discovered the quartic surface with sixteen nodal points, which is called after him. His reputation is primarily based on his introduction of the "ideal" numbers in the theory of algebraic domains of rationality (1846). This theory was inspired partly by Kummer's attempts to prove Fermat's great theorem and

[1]D. Hilbert, *Ueber das Unendliche,* Mathematische Annalen 95 (1926) pp. 161-190; French translation, Acta Mathematica 48 (1926) pp. 91-122.

partly by Gauss' theory of biquadratic residues in which the conception of prime factors had been introduced within the domain of complex numbers. Kummer's "ideal" factors allowed unique decomposition of numbers into prime factors in a general domain of rationality. This discovery made possible great advances in the arithmetic of algebraic numbers, which were later masterfully summarized in the report of David Hilbert written for the German mathematical society in 1897. The theory of Dedekind and Weber, which established a relation between the theory of algebraic functions and the theory of algebraic numbers in a certain domain of rationality (1882), was an example of the influence of Kummer's theory on the process of arithmetisation of mathematics.

Leopold Kronecker, a man of private means, settled in Berlin in 1855 where he taught for many years at the university without a formal professional chair, only accepting one after Kummer's retirement in 1883. Kronecker's main contributions were in elliptic functions, in ideal theory, and in the arithmetic of quadratic forms; his published lectures on the theory of numbers are careful expositions of his own and of previous discoveries, and also show clearly his belief in the necessity of the arithmetisation of mathematics. This belief was based on his search for rigor; mathematics, he thought, should be based on number, and all number on the natural number. The number π, for instance, rather than be derived in the usual geometrical way, should be based on the series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ and thus on a combination of integer numbers; certain continued fractions for π might also serve the same purpose. Kronecker's endeavor to force everything mathematical into the pattern of number theory is illustrated by his well-known statement at a meeting in Berlin in 1886: "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk."[1] He accepted a definition of a mathematical entity only in the case that it can be verified in a finite number of steps. Thus he coped with the difficulty of the actually infinite by refusing to accept it. Plato's slogan that God always "geometrizes" was replaced, in Kronecker's school, by the slogan that God always "arithmetizes."

Kronecker's teaching on the actually infinite was in flagrant contrast to the theories of Dedekind and especially of Cantor. Richard Dedekind, for thirty-one years professor at the Technische Hochschule in Brunswick, constructed a rigorous theory of the irrational. In two small books, "Stetigkeit und Irrationalzahlen" (1872) and "Was sind und was sollen die Zahlen" (1882)[2], he accomplished for modern mathematics what Eudoxos had done for Greek mathematics. There is a great similarity between the "Dedekind cut" with which modern mathematics (except the Kronecker school) defines irrational numbers and the ancient Eudoxos theory as presented in the fifth book of Euclid's elements. Cantor and Weierstrass gave arithmetical definitions of the irrational numbers differing somewhat from

[1] "The integer numbers have been made by God, everything else is the work of man."

[2] Translated as "Continuity and Irrational Numbers", "The Nature and the Meaning of Numbers" by A. Beman (Chicago, 1901).

GEORG CANTOR (1845–1918)

the Dedekind theory but based on similar considerations.

The greatest heretic in Kronecker's eye, however, was Georg Cantor. Cantor, who taught at Halle from 1869 until 1905, is known not only because of his theory of the irrational number, but also because of his theory of aggregates ("Mengenlehre"). With this theory Cantor created an entirely new field of mathematical research, which was able to satisfy the most subtle demands of rigor once its premises were accepted. Cantor's publications began in 1870 and continued for many years; in 1883 he published his "Grundlagen einer allgemeinen Mannigfaltigkeitslehre." In these papers Cantor developed a theory of transfinite cardinal numbers based on a systematical mathematical treatment of the actually infinite. He assigned the lowest transfinite cardinal number $\aleph$ to a denumerable set, giving the continuum a higher transfinite number, and it thus became possible to create an arithmetic of transfinite numbers analogous to ordinary arithmetic. Cantor also defined transfinite ordinal numbers, expressing the way in which infinite sets are ordered.

These discoveries of Cantor were a continuation of the ancient scholastic speculations on the nature of the infinite, and Cantor was well aware of it. He defended St. Augustine's full acceptance of the actually infinite, but had to defend himself against the opposition of many mathematicians who refused to accept the infinite except as a process expressed by ∞. Cantor's leading opponent was Kronecker, who represented a totally opposite tendency in the same process of arithmetization of mathematics. Cantor finally won full acceptance

when the enormous importance of his theory for the foundation of real function theory and of topology became more and more obvious—this especially after Lebesgue in 1901 had enriched the theory of aggregates with his theory of measure. There remained logical difficulties in the theory of transfinite numbers and paradoxes appeared, such as those of Burali Forti and Russell. This again led to different schools of thought on the foundation of mathematics. The Twentieth Century controversy between the formalists and the intuitinists was a continuation on a novel level of the controversy between Cantor and Kronecker.

16. Contemporaneous with this remarkable development of algebra and analysis was the equally remarkable flowering of geometry. It can be traced back to Monge's instruction, in which we find the roots of both the "synthetic" and the "algebraic" method in geometry. In the work of Monge's pupils both methods became separated, the "synthetic" method developed into projective geometry, the "algebraic" method into our modern analytical and algebraic geometry. Projective geometry as a separate science began with Poncelet's book of 1822. There were priority difficulties, as so often in cases concerning a fundamental discovery, since Poncelet had to face the rivalry of Joseph Gergonne, professor at Montpelier. Gergonne published several important papers on projective geometry in which he grasped the meaning of duality in geometry simultaneously with Poncelet. These papers appeared in the "Annales de mathématiques," the first purely mathematical periodical. Gergonne was its editor; it appeared from 1810 to 1832.

Typical of Poncelet's mode of thinking was another principle, that of continuity, which enabled him to derive the properties of one figure from those of another. He expressed the principle as follows:

> If the figure results from another by a continuous change, and is as general as the first, then a property proved on the first figure can be transferred to the other without further consideration.

This was a principle which had to be handled with great care, since the formulation was far from precise. Only modern algebra has been able to define its domain more accurately. In the hands of Poncelet and his school it led to interesting, new, and accurate results, especially when it was applied to changes from the real to the imaginary. It thus enabled Poncelet to state that all circles in the plane had "ideally two imaginary points at infinity in common," which also brought in the so-called "line at infinity" of the plane. G. H. Hardy has remarked that this means that projective geometry accepted the actually infinite without any scruples.[1] The analysts were to remain divided on this subject.

Poncelet's ideas were further developed by German geometers. In 1826 appeared the first of Steiner's publications, in 1827 Möbius' "Barycentrischer Calcül," in 1828 the first volume of Plücker's "Analytisch-geometrische Entwicklungen." In 1831 appeared the second

[1] G. H. Hardy, *A Course of Pure Mathematics* (Cambridge 1933, 6th ed.) Appendix IV.

volume, followed in 1832 by Steiner's "Systematische Entwicklung." The last of the great German pioneer works in this type of geometry appeared in 1847 with the publication of Von Staudt's axiomatic "Geometrie der Lage."

Both the synthetic and the algebraic approach to geometry were represented among these German geometers. The typical representative of the synthetic (or "pure") school was Jacob Steiner, a self-made Swiss farmer's son, a "Hirtenknabe", who became enamored of geometry by making the acquaintance of Pestalozzi's ideas. He decided to study at Heidelberg and later taught at Berlin, where from 1834 until his death in 1863 he held a chair at the university. Steiner was thoroughly a geometer; he hated the use of algebra and analysis to such an extent that he even disliked figures. Geometry in his opinion could best be learned by concentrated thought. Calculating, he said, replaces, while geometry stimulates, thinking. This was certainly true for Steiner himself, whose methods have enriched geometry with a large number of beautiful and often intricate theorems. We owe him the discovery of the Steiner surface with a double infinity of conics on it (also called the Roman surface). He often omitted the proof of his theorems, which has made Steiner's collected works a treasure trove for geometers in search of problems to solve.

Steiner constructed his projective geometry in a strictly systematic way, passing from perspectivity to projectivity and from there to the conic sections. He also solved a number of isoperimetrical problems in his own typical geometrical way. His proof (1836) that the circle is the figure of largest area for all closed curves

JAKOB STEINER (1796–1863)

of given perimeter made use of a procedure by which every figure of given perimeter which is not a circle could be changed into another one of the same perimeter and of larger area. Steiner's conclusion that the circle therefore represented the maximum suffered from one omission: he did not prove the actual existence of a maximum. Dirichlet tried to point it out to Steiner; a rigorous proof was later given by Weierstrass.[1]

Steiner still needed a metric to define the cross ratio of four points or lines. This defect in the theory was removed by Christian Von Staudt, for many years a professor at the University of Erlangen. Von Staudt, in his "Geometrie der Lage," defined the "Wurf" of four points on a straight line in a purely projective way, and then showed its identity with the cross ratio. He used for this purpose the so-called Moebius net construction, which leads to axiomatic considerations closely related to Dedekind's work when irrational values of projective coordinates are introduced. In 1857 Von Staudt showed how imaginary elements can be rigorously introduced into geometry as double elements of elliptic involutions.

During the next decades synthetic geometry grew greatly in content on the foundations laid by Poncelet, Steiner, and Von Staudt. It was eventually made the subject of a number of standard textbooks, of which Reye's "Geometrie der Lage" (1868, 3rd ed. 1886-1892)[2] is one of the best known examples.

17. Representatives of algebraic geometry were

[1]W. Blaschke, *Kreis und Kugel* (Leipzig, 1916) pp. 1-42.

[2]Translated as "Lectures on the geometry of position" (New York, 1898).

Moebius and Plücker in Germany, Chasles in France, and Cayley in England. August Ferdinand Moebius, more than fifty years observer, later director of the Leipzig astronomical observatory, was a scientist of many parts. In his book "Der barycentrische Calcül" he was the first to introduce homogeneous coordinates. When the masses m_1, m_2, m_3 are placed at the vertices of a fixed triangle, Moebius gave to the center of gravity (barycentrum) of these masses the coordinates $m_1 : m_2 : m_3$, and showed how these coordinates are well fitted to describe the projective and affine properties of the plane. Homogeneous coordinates, from now on, became the accepted tool for the algebraic treatment of projective geometry. Working in quiet isolation not unlike his contemporary Von Staudt, Moebius made many other interesting discoveries. An example is the null system in the theory of line congruences, which he introduced in his textbook on statics (1837). The "Moebius strip," a first example of a non-orientable surface, is a reminder of the fact that Moebius is also one of the founders of our modern science of topology.

Julius Plücker, who taught for many years at Bonn, was an experimental physicist as well as a geometer. He made a series of discoveries in crystal magnetism, electrical conduction in gases, and spectroscopy. In a series of papers and books, especially in his "Neue Geometrie des Raumes" (1868-69) he reconstructed analytical geometry by the application of a wealth of new ideas. Plücker showed the power of the abbreviated notation, in which for instance $C_1 + \lambda C_2 = 0$ represents a pencil of conics. In this book he introduced homoge-

neous coordinates, now as "projective" coordinates based on a fundamental tetrahedron, and also the fundamental principle that geometry needs not solely be based on points as basic element. Lines, planes, circles, spheres can all be used as the elements ("Raumelemente") on which a geometry can be based. This fertile conception threw new light on both synthetic and algebraic geometry, and created new forms of duality. The number of dimensions of a particular form of geometry could now be any positive integer number, depending on the number of parameters necessary to define the "element." Plücker also published a general theory of algebraic curves in the plane, in which he derived the "Plücker relations" between the number of singularities (1834, 1839).

Michel Chasles, for many years the leading representative of geometry in France, was a pupil of the Ecole Polytechnique in the later days of Monge and in 1841 became professor at this institute. In 1846 he accepted the chair of higher geometry at the Sorbonne, especially established for him, where he taught for many years. Chasles' work had much in common with that of Plücker, notably in his ability to obtain the maximum of geometrical information from his equations. It led him to adroit operation with isotropic lines and circular points at infinity. Chasles followed Poncelet in the use of "enumerative" methods, which in his hands developed into a new branch of geometry, the so-called "enumerative geometry." This field was later fully explored by Hermann Schubert in his "Kalkül der abzählenden Geometrie" (1879) and by H. G. Zeuthen in his "Abzählende Methoden" (1914). Both books reveal the

strength as well as the weakness of this type of algebra in geometrical language. Its initial success provoked a reaction led by E. Study, who stressed that "precision in geometricis may not perpetually be treated as incidental."[1]

Chasles had a fine appreciation for the history of mathematics, especially of geometry. His well-known "Aperçu historique sur l'origine et le développement des méthodes en géométrie" (1837) stands at the beginning of modern history of mathematics. It is a very readable text on Greek and modern geometry, and is a good example of a history of mathematics written by a productive scientist.

18. During these years of almost feverish productivity in the new projective and algebraic geometries another novel and even more revolutionary type of geometry lay hidden in a few obscure publications discarded by most leading mathematicians. The question whether Euclid's parallel postulate is an independent axiom or can be derived from other axioms had puzzled mathematicians for two thousand years. Ptolemy had tried to find an answer in Antiquity, Nasīr al-dīn in the Middle Ages, Lambert and Legendre in the Eighteenth Century. All these men had tried to prove the axiom and had failed, even if they reached some very interesting results in the course of their investigation. Gauss was the first man who believed in the independent nature of the parallel postulate, which implied that other geometries, based on another choice of axiom, were logically

[1]See E. Study, Verhandlungen Third Intern. Congress Heidelberg 1905, pp. 388-395, B. L. Van der Waerden, Diss. Leiden 1926.

Courtesy of Scripta Mathematica

NICOLAI IVANOVITCH LOBACHEVSKY (1793–1856)

possible. Gauss never published his thoughts on this subject. The first to challenge openly the authority of two millennia and to construct a non-euclidean geometry were a Russian, Nikolai Ivanovitch Lobachevsky, and a Hungarian, Janos Bolyai. The first in time to publish his idea was Lobachevsky, who was a professor in Kazan and lectured on the subject of Euclid's parallel axiom in 1826. His first book appeared in 1829-30 and was written in Russian. Few people took notice of it. Even a later German edition with the title "Geometrische Untersuchungen zur Theorie der Parallellinien" received little attention, even though Gauss showed interest. By that time Bolyai had already published his ideas on the subject.

Janos (Johann) Bolyai was the son of a mathematics teacher in a provincial town of Hungary. This teacher, Farkas (Wolfgang) Bolyai, had studied at Göttingen when Gauss was also a student there. Both men kept up an occasional correspondence. Farkas spent much time in trying to prove Euclid's fifth postulate (p. 60), but could not come to a definite conclusion. His son inherited his passion and also began to work on a proof despite his father's plea to do something else:

> "You should detest it just as much as lewd intercourse, it can deprive you of all your leisure, your health, your rest, and the whole happiness of your life. This abysmal darkness might perhaps devour a thousand towering Newtons, it will never be light on earth. . . ." (Letter of 1820).

Janos Bolyai entered the army and built up a reputation as a dashing officer. He began to accept Euclid's postulate as an independent axiom and discovered that

18

it was possible to construct a geometry, based on another axiom, in which through one point in a plane an infinity of lines can be laid which do not intersect a line in the plane. This was the same idea which had already occurred to Gauss and Lobachevsky. Bolyai wrote down his reflections, which were published in 1832 as an appendix to a book of his father and which had the title "Appendix scientiam spatii absolute veram exhibens." The worrying father wrote to Gauss for advice on the unorthodox views of his son. When the answer from Göttingen came, it contained enthusiastic approval of the younger Bolyai's work. Added to this was Gauss' remark that he could not praise Bolyai, since this would mean self-praise, the ideas of the "Appendix" having been his own for many years.

Young Janos was deeply disappointed by this letter of approval which elevated him to the rank of a great scientist but robbed him of his priority. His disappointment increased when he met with little further recognition. He became even more upset when Lobachevsky's book was published in German (1840); he never published any more mathematics.

Bolyai's and Lobachevsky's theories were similar in principle, though their papers were very different. It is remarkable how the new ideas sprang up independently in Göttingen, Budapest, and Kazan, and in the same period after an incubation period of two thousand years. It is also remarkable how they matured partly outside the geographical periphery of the world of mathematical research. Sometimes great new ideas are born outside, not inside, the schools.

Non-euclidean geometry (the name is due to Gauss) remained for several decades an obscure field of science. Most mathematicians ignored it, the prevailing Kantian philosophy refused to take it seriously. The first leading scientist to understand its full importance was Riemann, whose general theory of manifolds (1854) made full allowance not only for the existing types of non-euclidean geometry, but also for many other, so-called Riemannian, geometries. However, full acceptance of these theories came only when the generation after Riemann began to understand the meaning of his theories (1870 and later).

Still another generalization of classical geometry originated in the years before Riemann and did not find full appreciation until after his death. This was the geometry of more than three dimensions. It came fully developed into the world in Grassmann's "Ausdehnungslehre" ("Theory of Extension") of 1844. Hermann Grassmann was a teacher at the Gymnasium in Stettin and was a man of extraordinary versatility; he wrote on such varied subjects as electric currents, colors and acoustics, linguistics, botany, and folklore. His Sanskrit dictionary on the Rigveda is still in use. The "Ausdehnungslehre," of which a revised and more readable edition was published in 1861, was written in strictly euclidean form. It built up a geometry in a space of n dimensions, first in affine, then in metrical space. Grassmann used an invariant symbolism, in which we now recognize a vector and tensor notation (his "gap" products are tensors) but which made his work almost inaccessible to his contemporaries. A later generation took parts of Grassmann's structure to build up vector analysis for affine and for metrical spaces.

Although Cayley in 1843 introduced the same conception of a space of n dimensions in a far less forbidding form, geometry of more than three dimensions was received with distrust and incredulity. Here again Riemann's address of 1854 made full appreciation easier. Added to Riemann's ideas were those of Plücker, who pointed out that space elements need not be points (1865), so that the geometry of lines in three space could be considered as a four-dimensional geometry, or, as Klein has stressed, as the geometry of a four-dimensional quadric in a five-dimensional space. Full acceptance of geometries of more than three dimensions occurred only in the later part of the Nineteenth Century, mainly because of their use in interpreting the theory of algebraic and differential forms in more than three variables.

19. The names of Hamilton and Cayley show that by 1840 English speaking mathematicians had at last begun to catch up with their continental colleagues. Until well into the Nineteenth Century the Cambridge and Oxford dons regarded any attempt at improvement of the theory of fluxions as impious revolt against the sacred memory of Newton. The result was that the Newtonian school of England and the Leibnizian school of the continent drifted apart to such an extent that Euler, in his integral calculus (1768), considered a union of both methods of expressions as useless. The dilemma was broken in 1812 by a group of young mathematicians at Cambridge who, under the inspiration of the older Robert Woodhouse, formed an "Analytical Society" to propagate the differential notation.

ARTHUR CAYLEY (1821–1895)

Leaders were George Peacock, Charles Babbage, and John Herschel. They tried, in Babbage's words, to advocate "the principles of pure *d*-ism as opposed to the *dot*-age of the university." This movement met initially with severe criticism, which was overcome by such actions as the publication of an English translation of Lacroix' "Elementary Treatise on the Differential and Integral Calculus" (1816). The new generation in England now began to participate in modern mathematics.

The first important contribution came not from the Cambridge group, however, but from some mathematicians who had taken up continental mathematics independently. The most important of these mathematicians were Hamilton and George Green. It is interesting to notice that with both men, as well as with Nathaniel Bowditch in New England, the inspiration to study "pure *d*-ism" came from the study of Laplace's "Mécanique Céleste." Green, who was a self-taught miller's son of Nottingham, followed with great care the new discoveries in electricity. There was, at that time (c. 1825), almost no mathematical theory to account for the electrical phenomena; Poisson, in 1812, had made no more than a beginning. Green read Laplace, and, in his own words:

> "Considering how desirable it was that a power of universal agency, like electricity, should, as far as possible, be submitted to calculation, and reflecting on the advantages that arise in the solution of many difficult problems, from dispersing altogether with a particular examination of each of the forces which actuate the various bodies in any system, by confining the attention solely on that peculiar function on whose differentials they all depend, I was induced to try whether it would be possible to discover any general relations, existing between this function and the quantities of electricity in the bodies producing it."

The result was Green's "Essay on the Application of Mathematical Analysis to Theories of Electricity and Magnetism" (1828), the first attempt at a mathematical theory of electro-magnetism. It was the beginning of modern mathematical physics in England and, with Gauss' paper of 1839, established the theory of potential as an independent branch of mathematics. Gauss did not know Green's paper, which only became wider known when William Thompson (the later Lord Kelvin), had it reprinted in Crelle's Journal of 1846. Yet the kinship of Gauss and Green was so close that where Green selected the term "potential function", Gauss selected almost the same term, "potential," for the solution of Laplace's equation. Two closely related identities, connecting line and surface integrals, are called the formula of Green and the formula of Gauss. The term "Green's function" in the solution of partial differential equations also honors the miller's son who studied Laplace in his spare time.

We have no room for a sketch of the further development of mathematical physics in England, or, for that matter, in Germany. With this development the names of Stokes, Rayleigh, Kelvin, and Maxwell, of Kirchhoff and Helmholtz, of Gibbs and of many others are connected. These men contributed to the solution of partial differential equations to such an extent that mathematical physics and the theory of linear partial differential equations of the second order sometimes seemed to become identified. Mathematical physics, however,

brought fertile ideas to other fields of mathematics, to probability and complex function theory, as well as to geometry. Of particular importance was James Clerk Maxwell's "Treatise on Electricity and Magnetism" (1873, 2 vols.), which gave a systematic mathematical exposition of the theory of electromagnetism based on Faraday's experiments. This theory of Maxwell eventually dominated mathematical electricity, and later inspired the theories of Lorentz on the electron and Einstein on relativity.

20. Nineteenth Century pure mathematics in England was primarily algebra, with applications primarily to geometry and with three men, Cayley, Sylvester, and Salmon, leading in this field. Arthur Cayley devoted his early years to the study and practice of law, but in 1863 he accepted the new Sadlerian professorship of mathematics at Cambridge where he taught for thirty years. In the forties, while Cayley practiced law in London, he met Sylvester, at that time an actuary; and from those years dates Cayley's and Sylvester's common interest in the algebra of forms—or quantics, as Cayley called them. Their collaboration meant the beginning of the theory of algebraic invariants.

This theory had been "in the air" for many years, especially after determinants began to be a subject of study. The early work of Cayley and Sylvester went beyond mere determinants, it was a conscious attempt to give a systematic theory of invariants of algebraic forms, complete with its own symbolism and rules of composition. This was the theory which was later improved by Aronhold and Clebsch in Germany and

JAMES JOSEPH SYLVESTER (1814–1897)
From an old photograph

formed the algebraic counterpart of Poncelet's projective geometry. Cayley's voluminous work covered a large variety of topics in the fields of finite groups, algebraic curves, determinants, and invariants of algebraic forms. To his best known works belong his nine "Memoirs on Quantics" (1854-1878). The sixth paper of this series (1859) contained the projective definition of a metric with respect to a conic section. This discovery led Cayley to the projective definition of the euclidean metric and in this way enabled him to assign to metrical geometry its position inside the framework of projective geometry. The relation of this projective metric to non-euclidean geometry escaped the eye of Cayley; it was later discovered by Felix Klein.

James Joseph Sylvester was not only a mathematician but also a poet, a wit, and with Leibniz the greatest creator of new terms in the whole history of mathematics. From 1855 to 1869 he taught at Woolwich Military Academy. He was twice in America, the first time as a professor at the University of Virginia (1841-42), the second time as a professor at Johns Hopkins University in Baltimore (1877-1883). During this second period he was one of the first to establish graduate work in mathematics at American universities; with the teaching of Sylvester the flourishing of mathematics began in the United States.

Two of Sylvester's many contributions to algebra have become classics: his theory of elementary divisors (1851, rediscovered by Weierstrass in 1868) and his law of inertia of quadratic forms (1852, already known to Jacobi and Riemann, but not published). We also owe to Sylvester many terms now generally accepted,

such as invariant, covariant, contravariant, cogredient and syzygy. Many anecdotes have been attributed to him—several of the absent-minded-professor variety.

The third English algebrist-geometer was George Salmon, who during his long life was connected with Trinity College, Dublin, Hamilton's alma mater, where he instructed in both mathematics and divinity. His main merit lies in his well-known textbooks which excel in clarity and charm. These books opened the road to analytical geometry and invariant theory to several generations of students in many countries and even now have hardly been surpassed. They are the "Conic Sections" (1848), "Higher Plane Curves" (1852), "Modern Higher Algebra" (1859), and the "Analytic Geometry of Three Dimensions" (1862). The study of these books can still be highly recommended to all students of geometry.

21. Two products of the algebra of the United Kingdom deserve our special attention: Hamilton's quaternions and Clifford's biquaternions. Hamilton, the Royal Astronomer of Ireland, having completed his work on mechanics and optics, turned in 1835 to algebra. His "Theory of Algebraic Couples" (1835) defined algebra as the science of pure time and constructed a rigorous algebra of complex numbers on the conception of a complex number as a number pair. This was probably independent of Gauss, who in his theory of biquadratic residues (1831) had also constructed a rigorous algebra of complex numbers, but based on the geometry of the complex plane. Both conceptions are now equally accepted. Hamilton subsequently tried to penetrate

W. R. Hamilton

(1805–1865)

into the algebra of number triples, number quadruples, etc. The light dawned upon him—as his admirers like to tell—on a certain October day of 1843, when walking under a Dublin bridge he discovered the quaternion. His investigations on quaternions were published in two big books, the "Lectures on Quaternions" (1853) and the posthumous "Elements of Quaternions" (1866). The best known part of this quaternion calculus was the theory of vectors (the name is due to Hamilton), which was also part of Grassmann's theory of extension. It is mainly because of this fact that the algebraic works of Hamilton and Grassmann are now frequently quoted. In Hamilton's days, however, and long afterwards, the quaternions themselves were the subject of an exaggerated admiration. Some British mathematicians saw in the calculus of quaternions a kind of Leibnizian "arithmetica universalis," which of course aroused a reaction (Heaviside versus Tait) in which quaternions lost much of their glory. The theory of hypercomplex numbers, elaborated by Peirce, Study, Frobenius, and Cartan, has eventually placed quaternions in their legitimate place as the simplest associative number system of more than two units. The cult of the quaternion in its heyday even led to an "International Association for Promoting the Study of Quaternions and Allied Systems of Mathematics," which disappeared as a victim of the World War I. Another aspect of the quaternion controversy was the fight between partisans of Hamilton and Grassmann, when, through the efforts of Gibbs in America and Heaviside in England, vector analysis had emerged as an independent brand of mathematics. This controversy raged between 1890

and the first world war and was finally solved by the application of the theory of groups, which established the merits of each method in its own field of operation.[1]

William Kingdon Clifford, who died in 1879 at the age of thirty-three, taught at Trinity College, Cambridge and at University College, London. He was one of the first Englishmen who understood Riemann, and with him shared a deep interest in the origin of our space conceptions. Clifford developed a geometry of motion for the study of which he generalized Hamilton's quaternions into the so-called biquaternions (1873-1876). These were quaternions with coefficients taken from a system of complex numbers $a + b\epsilon$, where ϵ^2 may be $+1$, -1 or 0, and which could also be used for the study of motion in non-euclidean spaces. Clifford's "Common Sense in the Exact Sciences" is still good reading; it brings out the kinship in thinking between him and Felix Klein. This kinship is also revealed in the term "spaces of Clifford-Klein" for certain closed euclidean manifolds in non-euclidean geometry. If Clifford had lived, Riemann's ideas might have influenced British mathematicians a generation earlier than they actually did.

For many decades pure mathematics in the English speaking countries maintained its strong emphasis on formal algebra. It influenced the work of Benjamin Peirce of Harvard University, a pupil of Nathaniel Bowditch, who did distinguished work in celestial mechanics and in 1872 published his "Linear Associative Algebras," one of the first systematic studies of hypercomplex numbers. The formalist trend in English mathematics may also account for the appearance of an investigation of "The Laws of Thought" (1854) by George Boole of Queen's College, Dublin. Here it was shown how the laws of formal logic, which had been codified by Aristotle and taught for centuries in the universities, could themselves be made the subject of a calculus. It established principles in harmony with Leibniz' idea of a "characteristica generalis." This "algebra of logic" opened a school of thought which endeavored to establish a unification of logic and mathematics. It received its impetus from Gottlob Frege's book "Die Grundlagen der Arithmetik" (1884), which offered a derivation of arithmetical concepts from logic. These investigations reached a climax in the Twentieth Century with the "Principia Mathematica" of Bertrand Russell and Alfred N. Whitehead (1910-1913); they also influenced the later work of Hilbert on the foundations of arithmetic and the elimination of the paradoxes of the infinite.[1]

[1]F. Klein, *Vorlesungen über die Entwicklung der Mathematik in 19. Jahrhundert* (Berlin, 1927) II, pp. 27-52; J. A. Schouten, *Grundlagen der Vektor-und Affinoranalysis* (Leipzig, 1914).

22. The papers on the theory of invariants by Cayley and Sylvester received the greatest attention in Germany, where several mathematicians developed the theory into a science based on a complete algorithm. The main figures were Hesse, Aronhold, Clebsch, and Gordan. Hesse, who was a professor at Königsberg and later at Heidelberg and Münich, showed, like Plücker,

[1]D. Hilbert–W. Ackermann, *Grundzüge der theoretischen Logik* (Berlin, 1928). M Black, *The Nature of Mathematics* (New York, London, 1933).

the power of abbreviated methods in analytical geometry. He liked to reason with the aid of homogeneous coordinates and of determinants. Aronhold, who taught at the Technische Hochschule in Berlin, wrote a paper in 1858 in which he developed a consistent symbolism in invariant theory with the aid of so-called "ideal" factors (which bear no relation to those of Kummer); this symbolism was further developed by Clebsch in 1861, under whose hands the "Clebsch-Aronhold" symbolism became the almost universally accepted method for the systematic investigation of algebraic invariants. We now recognize in this symbolism as well as in Hamilton's vectors, Grassman's gap products, and Gibbs' dyadics, special aspects of tensor algebra. This theory of invariants was later enriched by Paul Gordan of the University of Erlangen, who proved (1868-69) that to every binary form belongs a finite system of rational invariants and covariants in which all other rational invariants and covariants can be expressed in rational form. This theorem of Gordan (the "Endlichkeitssatz") was extended by Hilbert in 1890 to algebraic forms in n variables.

Alfred Clebsch was professor at Karlsruhe, Giessen, and Göttingen and died at thirty-nine years of age. His life was a condensation of remarkable achievements. He published a book on elasticity (1862), following the leadership of Lamé and De Saint Venant in France; he applied his theory of invariants to projective geometry. He was one of the first men who understood Riemann and was a founder of that branch of algebraic geometry in which Riemann's theory of functions and of multiply connected surfaces was applied to real algebraic curves.

Clebsch-Gordan's "Theorie der Abelschen Funktionen" (1866) gave a broad outline of these ideas. Clebsch also founded the "Mathematische Annalen," for more than sixty years the leading mathematical journal. His lectures on geometry, published by F. Lindemann, remain a standard text on projective geometry.

23. By 1870 mathematics had grown into an enormous and unwieldy structure, divided into a large number of fields in which only specialists knew the way. Even great mathematicians—Hermite, Weierstrass, Cayley, Beltrami—at most could be proficient in only a few of these many fields. This specialization has constantly grown until at present it has reached alarming proportions. Reaction against it has never stopped, and some of the most important achievements of the last hundred years have been the result of a synthesis of different domains of mathematics.

Such a synthesis had been realized in the Eighteenth Century by the works of Lagrange and Laplace on mechanics. They remained a basis for very powerful work of varied character. The Nineteenth Century added to this new unifying principles, notably the theory of groups and Riemann's conception of function and of space. Their meaning can best be understood in the work of Klein, Lie, and Poincaré.

Felix Klein was Plücker's assistant in Bonn during the late sixties; it was here that he learned geometry. He visited Paris in 1870 when he was twenty-two years of age. Here he met Sophus Lie, a Norwegian six years his senior, who had become interested in mathematics only a short time before. The young men met the French

FELIX KLEIN (1849–1925)
From a photograph taken during his vigorous middle years

mathematicians, among them Camille Jordan of the Ecole Polytechnique and studied their work. Jordan, in 1870, had just written the "Traité des substitutions," his book on substitution groups and Galois' theory of equations. Klein and Lie began to understand the central importance of group theory and subsequently divided the field of mathematics more or less into two parts. Klein, as a rule, concentrated on discontinuous, Lie on continuous groups.

In 1872 Klein became professor at Erlangen. In his inaugural address he explained the importance of the group conception for the classification of the different fields of mathematics. The address, which became known as the "Erlangen program," declared every geometry to be the theory of invariants of a particular transformation group. By extending or narrowing the group we can pass from one type of geometry to another. Euclidean geometry is the study of the invariants of the metrical group, projective geometry of those of the projective group. Classification of groups of transformation gives us a classification of geometry; the theory of algebraic and differential invariants of each group gives us the analytical structure of a geometry. Cayley's projective definition of a metric allows us to consider metrical geometry in the frame of projective geometry. "Adjunction" of an invariant conic to a projective geometry in the plane gives us the non-euclidean geometries. Even relatively unknown topology received its proper place as the theory of invariants of continuous point transformations.

In the previous year Klein had given an important example of his mode of thinking when he showed how

non-euclidean geometries can be conceived as projective geometries with a Cayley metric. This brought full recognition at last to the neglected theories of Bolyai and Lobachevsky. Their logical consistency was now established. If there were logical mistakes in non-euclidean geometry, then they could be detected in projective geometry, although few mathematicians were willing to admit such a heresy. Later this idea of an "image" of one field of mathematics on another field was often used and played an important factor in Hilbert's axiomatics of geometry.

The theory of groups made possible a synthesis of the geometrical and algebraic work of Monge, Poncelet, Gauss, Cayley, Clebsch, Grassmann, and Riemann. Riemann's theory of space, which had offered so many suggestions of the Erlangen program, inspired not only Klein but also Helmholtz and Lie. Helmholtz in 1868 and 1884 studied Riemann's conception of space, partly by looking for a geometrical image of his theory of colors, partly by inquiring into the origin of our ocular measure. This led him to investigate the nature of geometrical axioms and especially Riemann's quadratic measurement. Lie improved on Helmholtz' speculations concerning the nature of Riemann's measurement by analyzing the nature of the underlying groups of transformations (1890). This "Lie-Helmholtz" space problem has been of importance not only to relativity and group theory, but also to physiology.

Klein gave an exposition of Riemann's conception of complex functions in his booklet "Ueber Riemann's Theorie der algebraischen Funktionen" (1882), in which he stressed how physical considerations can influence

MARIUS SOPHUS LIE (1842–1899)

even the subtlest type of mathematics. In the "Vorlesungen ueber das Ikosaeder" (1884) he showed that modern algebra could teach many new and surprising things about the ancient Platonic bodies. This work was a study of rotation groups of the regular bodies and their role as Galois groups of algebraic equations. In extensive studies by himself and by scores of pupils Klein applied the group conception to linear differential equations, to elliptic modular functions, to Abelian and to the new "automorphic" functions, the last in an interesting and friendly competition with Poincaré. Under Klein's inspiring leadership Göttingen, with its traditions of Gauss, Dirichlet, and Riemann, became a world center of mathematical research where young men and women of many nations gathered to study their special subjects as an integral part of the whole of mathematics. Klein gave inspiring lectures, the notes of which circulated in mimeographed form and provided whole generations of mathematicians with specialized information and—above all—with an understanding of the unity of their science. After Klein's death in 1925 several of these lecture notes were published in book form.

While in Paris Sophus Lie had discovered the contact transformation, and with this the key to the whole of Hamiltonian dynamics as a part of group theory. After his return to Norway he became a professor in Christiania, later, from 1886 to 1898, he taught at Leipzig. He devoted his whole life to the systematic study of continuous transformation groups and their invariants, demonstrating their central importance as a classifying principle in geometry, mechanics, ordinary and partial

differential equations. The result of this work was codified in a number of standard tomes, edited with the aid of Lie's pupils Scheffers and Engel ("Transformationsgruppen," 1888-1893; "Differentialgleichungen," 1891; "Kontinuierliche Gruppen," 1893; "Berührungstransformation," 1896). Lie's work has since been considerably enriched by the French mathematician, Elie Cartan.

24. France, faced with the enormous growth of mathematics in Germany, continued to produce excellent mathematicians in all fields. It is interesting to compare German and French mathematicians; Hermite with Weierstrass, Darboux with Klein, Hadamard with Hilbert, Paul Tannery with Moritz Cantor. From the forties to the sixties the leading mathematician was Joseph Liouville, professor at the Collège de France in Paris, a good teacher and organizer and editor for many years of the "Journal de mathématiques pures et appliquées." He investigated in a systematic way the arithmetic theory of quadratic forms of two and more variables, but "Liouville's theorem" in statistical mechanics shows him as a productive worker in an entirely different field. He established the existence of transcendental numbers and in 1844 proved that neither e nor e^2 can be a root of a quadratic equation with rational coefficients. This was a step in the chain of arguments which led from Lambert's proof in 1761 that π is irrational to Hermite's proof that e is transcendental (1873) and the final proof by F. Lindemann (a Weierstrass pupil) that π is transcendental (1882). Liouville and

several of his associates developed the differential geometry of curves and surfaces; the formulas of Frenet-Serret (1847) came out of Liouville's circle.

Charles Hermite, a professor at the Sorbonne and at the Ecole Polytechnique, became the leading representative of analysis in France after Cauchy's death in 1857. Hermite's work, as well as that of Liouville, was in the tradition of Gauss and Jacobi; it also showed kinship with that of Riemann and Weierstrass. Elliptic functions, modular functions, theta-functions, number and invariant theory all received his attention, as the names "Hermitian numbers", "Hermitian forms" testify. His friendship with the Dutch mathematician Stieltjes, who through Hermite's intervention obtained a chair at Toulouse, was a great encouragement to the discoverer of the Stieltjes integral and the application of continued fractions to the theory of moments. The appreciation was mutual: "Vous avez toujours raison et j'ai toujours tort,"[1] Hermite once wrote to his friend. The four volume "Correspondance" (1905) between Hermite and Stieltjes contains a wealth of material, mainly on functions of a complex variable.

The French geometrical tradition was gloriously continued in the books and papers of Gaston Darboux. Darboux was a geometer in the sense of Monge, approaching geometrical problems with full mastery of groups and differential equations, and working on problems of mechanics with a lively space intuition. Darboux was professor at the Collège de France and for half a century active in teaching. His most influential work was his standard "Leçons sur la théorie générale

[1] "You are always right and I am always wrong."

T. J. STIELTJES (1856–1894)

des surfaces" (4 vols., 1887-1896), which presented the results of a century of research in the differential geometry of curves and surfaces. In Darboux' hands this differential geometry became connected in the most varied ways with ordinary and partial differential equations as well as with mechanics. Darboux, with his administrative and pedagogical skill, his fine geometrical intuition, his mastery of analytical technique, and his understanding of Riemann, occupied a position in France somewhat analogous to that of Klein in Germany.

This second part of the Nineteenth Century was the period of the great French comprehensive textbooks on analysis and its applications, which often appeared under the name of "Cours d'analyse" and were written by leading mathematicians. The most famous are the "Cours d'analyse" of Camille Jordan (3 vols., 1882-87) and the "Traité d'analyse" of Emile Picard (3 vols., 1891-96), to which was added the "Cours d'analyse mathématique" by Edouard Goursat (2 vols., 1902-05).

25. The greatest French mathematician of the second half of the Nineteenth Century was Henri Poincaré, from 1881 until his death professor at the Sorbonne in Paris. No mathematician of his period commanded such a wide range of subjects and was able to enrich them all. Each year he lectured on a different subject; these lectures were edited by students and cover an enormous field: potential theory, light, electricity, conduction of heat, capillarity, electromagnetics, hydrodynamics, celestial mechanics, thermodynamics, probability. Every one of these lectures was brilliant in its own way;

HENRI POINCARÉ (1854–1912)

together they present ideas which have borne fruit in the works of others while many still await further elaboration. Poincaré, moreover, wrote a number of popular and semi-popular works which helped to give a general understanding of the problems of modern mathematics. Among them are "La valeur de la science" (1905) and "La science et l'hypothèse" (1906). Apart from these lectures Poincaré published a large number of papers on the so-called automorphic and fuchsian functions, on differential equations, on topology, and on the foundations of mathematics, treating with great mastery of technique and full understanding all pertinent fields of pure and applied mathematics. No mathematician of the Nineteenth Century, with the possible exception of Riemann, has so much to say to the present generation.

The key to the understanding of Poincaré's work may lie in his meditations on celestial mechanics, and in particular on the three-body problem ("Les méthodes nouvelles de mécanique céleste", 3 vols., 1893). Here he showed direct kinship with Laplace and demonstrated that even at the end of the Nineteenth Century the ancient mechanical problems concerning the universe had lost nothing of their pertinence to the productive mathematician. It was in connection with these problems that Poincaré studied divergent series and developed his theory of asymptotic expansions, that he worked on integral invariants, the stability of orbits, and the shape of celestial bodies. His fundamental discoveries on the behavior of the integral curves of differential equations near singularities, as well as in the large,

are related to his work on celestial mechanics. This is also true of his investigations on the nature of probability, another field in which he shared Laplace's interest. Poincaré was like Euler and Gauss, wherever we approach him we discover the stimulation of originality. Our modern theories concerning relativity, cosmogony, probability, and topology are all vitally influenced by Poincaré's work.

26. The Risorgimento, the national rebirth of Italy, also meant the rebirth of Italian mathematics. Several of the founders of modern mathematics in Italy participated in the struggles which liberated their country from Austria and unified it; later they combined political positions with their professional chairs. The influence of Riemann was strong, and through Klein, Clebsch, and Cayley the Italian mathematicians obtained their knowledge of geometry and the theory of invariants. They also became interested in the theory of elasticity with its strong geometrical appeal.

Among the founders of the new Italian school of mathematicians were Brioschi, Cremona, and Betti. In 1852 Francesco Brioschi became professor in Pavia, and in 1862 organized the technical institute at Milan where he taught until his death in 1897. He was a founder of the "Annali di matematica pura et applicata" (1858)—which indicated in the title its desire to emulate Crelle's and Liouville's journals. In 1858 in company with Betti and Casorati he visited the leading mathematicians of France and Germany. Volterra later claimed that "the scientific existence of Italy as a

nation" dated from this journey.[1] Brioschi was the Italian representative of the Cayley-Clebsch type of research in algebraic invariants. Luigi Cremona, after 1873 director of the engineering school in Rome, has given his name to the birational transformation of plane and space, the "Cremona" transformations (1863-65). He was also one of the originators of graphical statics.

Eugenio Beltrami was a pupil of Brioschi and occupied chairs in Bologna, Pisa, Pavia, and Rome. His main work in geometry was done between 1860 and 1870 when his differential parameters introduced a calculus of differential invariants into surface theory. Another contribution of that period was his study of so-called pseudospherical surfaces, which are surfaces of which the Gaussian curvature is negative constant. On such a pseudosphere we can realize a two-dimensional non-euclidean geometry of Bolyai. This was, with Klein's projective interpretation, a method to show that there were no internal contradictions in non-euclidean geometry, since such contradictions would also appear in ordinary surface theory.

By 1870 Riemann's ideas became more and more the common good of the younger generation of mathematicians. His theory of quadratic differential forms was made the subject of two papers by the German mathematicians E. B. Christoffel and R. Lipschitz (1870). The first paper introduced the "Christoffel" symbols. These investigations, combined with Beltrami's theory of differential parameters, brought Gregorio Ricci-Curbastro in Padua to his so-called absolute differential

[1]V. Volterra, Bull. Am. Math. Soc. 7 (1900) p. 60–62.

calculus (1884). This was a new invariant symbolism originally constructed to deal with the transformation theory of partial differential equations, but it provided at the same time a symbolism fitted for the transformation theory of quadratic differential forms.

In the hands of Ricci and of some of his pupils, notably of Tullio Levi-Civita, the absolute differential calculus developed into what we now call the theory of tensors. Tensors were able to provide a unification of many invariant symbolisms, and also showed their power in dealing with general theorems in elasticity, hydrodynamics, and relativity. The name tensor has its origin in elasticity (W. Voigt, 1900).

The most brilliant representative of differential geometry in Italy was Luigi Bianchi. His "Lezioni di geometria differenziale" (2nd ed., 3 vols., 1902-1909) ranks with Darboux' "Théorie générale des surfaces" as a classical exposition of Nineteenth Century differential geometry.

27. David Hilbert, professor at Göttingen, presented to the International Congress of Mathematicians in Paris in 1900 a series of twenty-three research projects. At that time Hilbert had already received recognition for his work on algebraic forms and had prepared his now famous book on the foundations of geometry ("Grundlagen der Geometrie," 1900). In this book he gave an analysis of the axioms on which euclidean geometry is based and explained how modern axiomatic research has been able to improve on the achievements of the Greeks.

In this address of 1900 Hilbert tried to grasp the trend of mathematical research of the past decades and to sketch the outline of future productive work.[1] A summary of his projects will give us a better understanding of the meaning of Nineteenth Century mathematics.

First of all Hilbert proposed the arithmetical formulation of the concept of continuum as it was presented in the works of Cauchy, Bolzano, and Cantor. Is there a cardinal number between that of a denumerable set and that of a continuum? and can the continuum be considered as a well ordered assemblage? Moreover, what can be said about the compatibility of the arithmetical axioms?

The next projects dealt with the foundations of geometry, with Lie's concept of a continuous group of transformations—is differentiability necessary?—and with the mathematical treatment of the axioms of physics.

Some special problems followed, first in arithmetic and in algebra. The irrationality or transcendence of certain numbers was still unknown (e.g. α^β for an algebraic α, and irrational β). Equally unknown was the proof of Riemann's hypothesis concerning the roots of the Zeta-function, as well as the formulation of the most general law of reciprocity in number theory. Another project in this field was the proof of the finiteness of certain complete systems of functions suggested by the theory of invariants.

[1]Translation in Bulletin Am. Math. Soc. (2), 8 (1901-02) pp. 437-479.

The fifteenth question demanded a rigorous formulation of Schubert's enumerative calculus, the sixteenth a study of the topology of algebraic curves and surfaces. Another problem concerned the division of space by congruent polyhedra.

The remaining projects dealt with differential equations and the calculus of variations. Are the solutions of regular problems in the calculus of variations always analytic? Has every regular variational problem under given boundary conditions a solution? How about the uniformization of analytic relations by means of automorphic functions? Hilbert ended his enumeration with an appeal for the further development of the calculus of variations.[1]

Hilbert's program demonstrated the vitality of mathematics at the end of the Nineteenth Century and contrasts sharply with the pessimistic outlook existing toward the end of the Eighteenth Century. At present some of Hilbert's problems have been solved; others still await their final solution. The development of mathematics in the years after 1900 has not disappointed the expectations raised at the close of the Nineteenth Century. Even Hilbert's genius, however, could not foresee some of the striking developments which actually have taken place and are taking place to-day. Twentieth Century mathematics has followed its own novel path to glory.

[1]A discussion of the problems outlined by Hilbert after thirty years in L. Bieberbach, *Über den Einfluss von Hilberts Pariser Vortrag über "Mathematische Probleme" auf die Entwicklung der Mathematik in den letzten dreissig Jahren,* Naturwissenschaften 18 (1936) pp. 1101-1111.

Literature.

The best history of Nineteenth Century mathematics is:

F. Klein, *Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert I, II* (Berlin, 1926, 1927).

A bibliography of leading Nineteenth Century mathematics in:

G. Sarton, *The Study of the History of Mathematics* (Cambridge, Mass, 1936) pp. 70-98.

Lists biographies of Abel, Adams, Airy, Appell, Aronhold, Babbage, Bachmann, Baire, Ball, Bellavitis, Beltrami, Bertrand, Bessel, Betti, Boltzmann, Bolyai, Bolzano, Boole, Borchardt, Brioschi, G. Cantor, L. Carnot, Cauchy, Cayley, Chasles, Chebyshev, Clausius, Clebsch, Clifford, Cournot, Couturat, Cremona, Darboux, G. Darwin, Dedekind, De Morgan, Dirichlet, Edgeworth, Eisenstein, Encke, Fiedler, Fourier, Fredholm, Frege, Fresnel, Fuchs, Galois, Gauss, Germain, Gibbs, Göpel, Gordan, Grassmann, Green, Halphen, Jevons, Jordan, Kelvin, Kirchhoff, Klein, Kovalevskaia, Kronecker, Kummer, Laguerre, Lamé, Laplace, Legendre, Lemoine, Leverrier, Lie, Liouville, Lobachevsky, Lorentz, MacCullagh, Maxwell, Méray, Minkowski, Mittag-Leffler, Möbius, F. Neumann, Newcomb, E. Noether, Olbers, Oppolzer, Painlevé, Peacock, B. Peirce, Pfaff, Plücker, Poincaré, Poinsot, Poisson, Poncelet, Ramanujan, Rankine, Rayleigh, Riemann, Rosenhain, Ruffini, Saint-Venant, Schwarz, Smith, v. Staudt, Steiner, Stokes, Sylow, Sylvester, Tait, Weierstrass. Other bibliographical material in the issues of Scripta Mathematica (N. Y., 1932—present).

The works of the following mathematicians have been collected: Abel. Beltrami, Betti, Bolzano (partly), Borchardt, Brioschi, G. Cantor, Cauchy, Cayley, Chebyshev, Clifford, Cremona, Dedekind, Dirichlet, Eisenstein, Fourier, Fuchs, Galois, Gauss, Gibbs, Grassmann, Green, Halphen, Hamilton, Hermite, Hilbert, Jacobi, Klein, Kronecker, Laguerre, Lie, Lobachevsky, G. A. Miller, Minkowski, Möbius, Plücker, Poincaré (partly), Ramanujan, Riemann, Ruffini, Schwarz, H. J. S. Smith, Steiner, Sylow, Sylvester, Tait, Weierstrass.

Furthermore:

L. de Launay, *Monge, Fondateur de l'Ecole Polytechnique* (Paris, 1933).

F. Klein u.a. *Materialien für eine wissenschaftliche Biographie von Gauss* (8 vols., Leipzig, 1911–1920).

Quaternion centenary celebration. Proc. Roy. Irish Acad. A50 (1945) pp. 69-98; among the articles is A. J. McConnell, *The Dublin mathematical school in the First Half of the Nineteenth Century.*

E. Kötter. *Die Entwicklung der synthetischen Geometrie von Monge bis auf von Staudt.* Jahresber. Deutsch. Math. Verein 5 (1901) pp. 1-486.

M. Black, *The Nature of Mathematics* (London, 1933), has a bibliography on symbolic logic.

V. F. Kagan, *Lobachevsky* (Moscow, Leningrad, 1945, in Russian).

D. J. Struik, *Outline of a History of Differential Geometry.* Isis 19 (1933) pp. 92-120; 20 (1934) pp. 161-191,

J. T. Merz, *A History of European Thought in the Nineteenth Century* (London, 1903-14).

J. Hadamard, *The Psychology of Invention in the Mathematical Field* (Princeton, N. J., 1945).

G. Prasad, *Some Great Mathematicians of the Nineteenth Century: Their Lives and Their Works* (2 vols. Benares, 1933–34).

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